UNIVERSITE IBN ZOHR

Ecole Nationale des Sciences Appliquées

Formation Doctorale : Sciences et Techniques de l'Ingénieur

Descipline : Mathématiques Appliquées

Spécialité : Calcul Stochastique et Systèmes Dynamiques

CONTRIBUTION A L'ETUDE DES SYSTEMES DYNAMIQUES HYBRIDES A COMMUTATIONS ALEATOIRES

Réalisé par: Chafai Imzegouan
Sous l'encadrement des Professeurs:
Hassane Bouzahir et Brahim Benaid

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1 Introduction

What is a Hybrid Systems ?

A hybrid system is a two-level system with the lower level governed by a set of modes described by differential equations and the upper level a coordinator that orchestrates the switching among the modes.
Clearly, the system admits continuous state that take values from a vector space and discrete states that take values from a discrete index set.
The interaction between the continuous and discrete states makes switching dynamical systems widely representative and complicatedly behaved.

Forced-free switched/jump dynamical system

$$dx(t) = f_r(x(t))dt, \qquad (1)$$

where $x \in \mathbb{R}^n$ is the continuous state, r is the discrete state taking values in a finite state space $\mathcal{M} = \{1, 2, ..., N\}$, and f_k, $k \in \mathcal{M}$, are vector fields.

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Random switching

We worked on random switching dynamic systems, these hybrid systems with a time-driven switching signal that fluctuates irregularly but obeys a distribution stochastically.

A well-known feasible set of random switching signals is the Markov jump where switches between different subsystems are governed by a finite-state Markov process/chain.

When the subsystems are linear and the switching is a Markov jump, the switched system is known to be a Markovian jump linear system.

Even when all the subsystems are deterministic, a random switching signal make the switched system random in nature, and the stability notions have to be defined in a stochastic manner.

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Motivation

Stability of solutions is important in applications such as communication networks, motor control, economic systems,..., and an important problem is to ensure stability.

A typical problem for switched systems goes as follows. It can be that all sub-systems of the switched system are stable but the switched system can be unstable.

Example 1.1

Consider the following randomly switched linear system :

$$dx(t) = A_{r(t)}x(t)dt \tag{2}$$

where $r(t)$ *is a continuous-time Markov chain taking values in a finite state space* $\mathcal{M} = \{1,2\}$ *with generator* $\mathcal{Q} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$*, and*

$$A_1 = \begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & 15 \\ 0 & -2 \end{pmatrix}$$

We can check that $dx(t) = A_1x(t)dt$ *and* $dx(t) = A_2x(t)dt$ *are both stable. However, System (2) is unstable.*

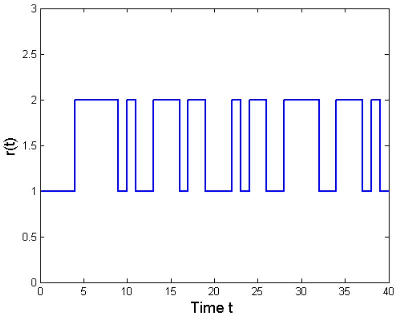

FIGURE: Jump process $r(t)$ with initial condition $r(0) = 1$

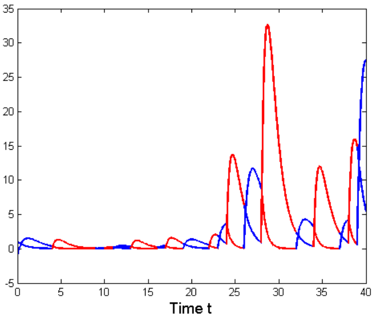

FIGURE: Trajectory of x as a function of time for System (2).

1.
2. **Mean Exponential Stability of Markovian Jump Linear System**

Consider the hybrid dynamical system with random switching as following :

$$dx(t) = A_{r(t)}x(t)dt, \quad x(0) = x_0 \in \mathbb{R}^n, \quad r(0) = r_0 \in \mathcal{M} \tag{3}$$

where $x(t)$ is the continuous state and $r(t)$ is a Markov process taking values in a finite state space $\mathcal{M} = \{1, 2, ..., N\}$ with generator $Q = (q_{ij})$, $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j \in \mathcal{M}} q_{ij} = 0$ for all $i \in \mathcal{M}$.

Its evolution is governed by the following probability transition :

$$P\{r(t+\Delta t) = j / r(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t) & i \neq j \\ 1 + q_{ii}\Delta t + o(\Delta t) & i = j \end{cases} \tag{4}$$

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Assume that the Markov chain $r(t)$ is irreducible in the sense that the system of equations

$$\begin{cases} \pi\mathcal{Q} = 0 \\ \pi\mathbb{1} = 1 \end{cases} \tag{5}$$

has a unique positive solution termed stationary distribution.
The process $(x(t), r(t))$ is associated with an infinitesimal operator $\mathcal{L}$ defined by :
For each $i \in \mathcal{M}$ and any $g(x,i) \in \mathcal{C}^1(\mathbb{R}^n)$

$$\mathcal{L}g(x,i) = \langle A_i x, \nabla g(x,i)\rangle + \mathcal{Q}g(x,.)(i) \tag{6}$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^n$.
$\nabla g(x,i)$ denotes the gradient (with respect to the variable x) of $g(x,i)$.
and $\mathcal{Q}g(x,.)(i) = \sum_{j\in\mathcal{M}} q_{ij} g(x,j)$.

Definition

For any initial condition (x_0, r_0), the equilibrium point $x = 0$ is said to be

- stochastically stable if there exists a positive constant $C(x_0, r_0)$ such that

$$E\Big[\int_0^\infty |x(t, x_0, r_0)|^2 dt\Big] \leq C(x_0, r_0), \tag{7}$$

- mean exponentially stable if there exist positive constants α and β such that

$$\mathbb{E}\Big[|x(t, x_0, r_0)|^2\Big] \leq \alpha |x_0| e^{-\beta t}. \tag{8}$$

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The symmetric and skew-symmetric part of a matrix $A \in \mathbb{R}^{n\times n}$ are expressed, respectively, as

$$A_s = \frac{1}{2}(A + A^T) \text{ and } A_u = \frac{1}{2}(A - A^T)$$

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Theorem 2.1

Assume that for any $i \in \mathcal{M}$, each matrix A_i is generalized definite negative, then, System (3) is mean exponentially stable.

- Consider the following Lyapunov function

$$V(x(t), r(t)) = |x(t)|^2.$$

- The infinitesimal operator acting on $V(x(t), i)$ is given by :

$$\begin{aligned}\mathcal{L}V(x(t), i) &= \langle A_i x(t), \nabla V(x(t), i)\rangle + \mathcal{Q}V(x(t), .)(i) \\ &\;\; . \\ &\;\; . \\ &\leq -\beta |x(t)|^2 = -\beta V(x(t), i) \end{aligned} \quad (9)$$

with $\beta = \sum_{i \in \mathcal{M}} \beta_i$ and $\beta_i = -\lambda_{max}(A_i + A_i^T) \geq 0$.

- By Dynkin's formula and Gronwall's inequality, we infer

$$\mathbb{E}\left[|x(t)|^2\right] \leq \alpha |x_0| e^{-\beta t}. \quad (10)$$

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Example 2.1

Consider System (3) with the following specifications. The Markov chain $r(t)$ *has four states, with generator* $Q = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$. *The stationary distribution of the Markov chain* $r(t)$ *is* $\pi = (0.25, 0.25, 0.25, 0.25)$, *which is obtained by solving Equation (5). The matrices are given by :*

$$A_1 = \begin{pmatrix} -2 & -1 \\ -2 & -3 \end{pmatrix}, \; A_2 = \begin{pmatrix} -1 & -2 \\ 2 & -5 \end{pmatrix}, \; A_3 = \begin{pmatrix} -3 & 3 \\ 0 & -1 \end{pmatrix}, \; A_4 = \begin{pmatrix} -2 & 0 \\ -1 & -1 \end{pmatrix}$$

The eigenvalues of $M_1 = A_1 + A_1^T$, $M_2 = A_2 + A_2^T$, $M_3 = A_3 + A_3^T$ *and* $M_4 = A_4 + A_4^T$ *are respectively* $(-8.1623 \quad -1.8377)$, $(-10 \quad -2)$, $(-7.6056 \quad -0.3944)$ *and* $(-4.4142 \quad -1.5858)$.
Then, System (3) is mean exponentially stable.

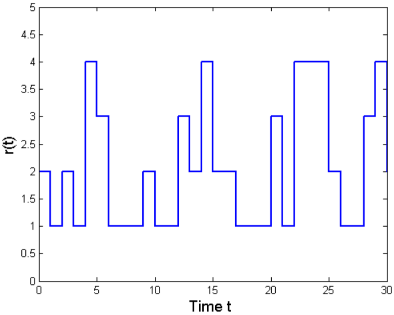

FIGURE: Markov jump $r(t)$ with initial condition $r_0 = 2$

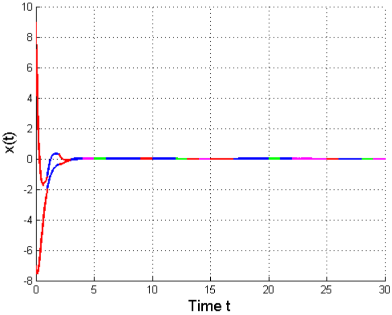

FIGURE: Solution curves of System (3) with initial condition $x_0 = [-7, 9]^T$

1.
2.
3. **Asymptotic Almost Sure Stability of Markovian Jump Linear System Associated with a Transfer Matrix**

Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{11}$$

associated with transfer matrix $G(s) = C(sI - A)^{-1}B + D$, with state feedback of the form

$$u_r(t) = \frac{1}{N}(1 - r(t))D^{-1}Cx(t)$$

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Schematic representation

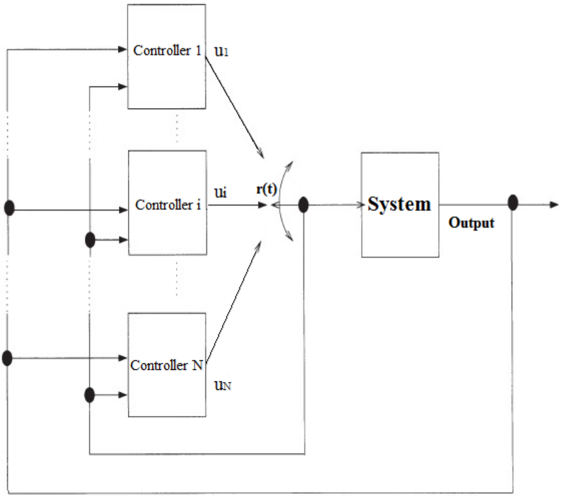

FIGURE: Dynamical system with Markovian switched controller

We rewrite (11) in following form :

$$\dot{x}(t) = A_{r(t)}x(t) \quad x(0) = x_0 \in \mathbb{R}^n, \quad r(0) = r_0 \in \mathcal{M} \tag{12}$$

with $A_{r(t)} = A + \frac{1}{N}(1 - r(t))BD^{-1}C$

Definition

A jump linear System (12) is said to be

- stochastically mean square stable if for any initial state x_0 and initial distribution ρ, we have

$$\int_0^{+\infty} \mathbb{E}_\rho\{|x(t;x_0,r_0)|^2\}dt < +\infty. \tag{13}$$

- asymptotically almost surely stable if for any initial state x_0 and initial distribution ρ, we have

$$P\{\lim_{t\to+\infty} |x(t;x_0,r_0)| = 0\} = 1. \tag{14}$$

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lemma (Shorten et al. 2014)

Given a Hurwitz matrix A, the symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with $D = D^T > 0$ is strictly positive real (SPR) if and only if $A(A - BD^{-1}C)$ has no real negative eigenvalue.

Lemma 3.2 (KYP)

Let A be Hurwitz, (A, B) be controllable, and (A, C) be observable. Then $G(s) = C(sI - A)^{-1}B + D$ is SPR if and only if there exist matrices $P = P^T > 0$, L and W, and a number $\alpha > 0$ satisfying

$$A^T P + PA + \alpha P = -L^T L$$
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Theorem 3.3

Assume that the transfer matrix $G(s)$ is SPR, with (A, B) controllable and (A, C) observable. Then the random switching System (12) *is asymptotically almost surely stable.*

- We define a Lyapunov function by the following expression :

$$V(x(t), r(t)) = x^T(t) P_{r(t)} x(t) \tag{15}$$

with $P_i = P_j = P = P^T > 0$.

- Next, by using KYP lemma, we show that for $i = 1$

$$\mathcal{L}V(x, 1) = -x^T(\alpha P_1 + L^T L)x < 0 \tag{16}$$

- For $i = \{2, ..., N\}$, by using KYP lemma, we show that

$$\mathcal{L}V(x, i) \leq -x^T\left[\alpha P + \frac{i-1}{N}(L - WD^{-1}C)^T(L - WD^{-1}C)\right]x \leq 0 \tag{17}$$

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- Using Dynkin's formula, we infer

$$\int_0^{+\infty} \mathbb{E}\left[|x(s)|^2\right] ds \leq C(x_0, r_0). \tag{18}$$

This means that the trivial solution of System (12) is stochastically mean square stable. By Lemma 3.1, System (12) is asymptotically almost surely stable.

Example 3.1

In this example, we consider $G(s)$ symmetric in order to verify easily that it is SPR.

Consider System (12) *associated to the symmetric transfer matrix* $G(s) = C(sI - A)^{-1}B + D$ *with*

$$A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -0.3 & -0.3 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The Markov jump process $r(t)$ takes values in $\mathcal{M} = \{1, 2, ..., 5\}$ with generator

$$Q = \begin{pmatrix} -4 & 0 & 1 & 1 & 2 \\ 1 & -2 & 0 & 1 & 0 \\ 6 & 1 & -8 & 0 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 2 & 1 & 1 & 1 & -5 \end{pmatrix}$$

The stationary distribution of irreducible Markov process $r(t)$ is $\pi = (0.27, 0.24, 0.09, 0.23, 0.17)$, which is obtained by solving Equation (5). Note that the five Hurwitz matrices associated to System (12) are given by

$$A_i = A + \frac{(1-i)}{5} B D^{-1} C \quad \text{for } i \in \{1, 2, ..., 5\}.$$

Then we have

$$A_1 = \begin{pmatrix} -2.0000 & -1.0000 \\ 1.0000 & 0.0000 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2.2350 & -1.6310 \\ 0.8330 & -0.3650 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -2.4700 & -2.2620 \\ 0.6660 & -0.7300 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -2.7050 & -2.8930 \\ 0.4990 & -1.0950 \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} -2.9400 & -3.5240 \\ 0.3320 & -1.4600 \end{pmatrix}.$$

Note that (A,B) and (A,C) are controllable and observable respectively $(rank([B,AB]) = 2$ and $rank([C^T, A^T C^T]) = 2)$.
The transfer function $G(s) = C(sI-A)^{-1}B + D$ is symmetric

$$G(s) = \frac{1}{(s^2+2.9s+2.425)} \begin{pmatrix} s^2+6.9s+9.535 & -s-1 \\ -s-1 & 2s^2+5.8s+4.7 \end{pmatrix}$$

and $A(A-BD^{-1}C)$ has no real negative eigenvalue, that means that $G(s)$ is SPR. Then, by Theorem 3.3 the hybrid System (12) is asymptotically almost surely stable.

$$A_3 = \begin{pmatrix} -2.4700 & -2.2620 \\ 0.6660 & -0.7300 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -2.7050 & -2.8930 \\ 0.4990 & -1.0950 \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} -2.9400 & -3.5240 \\ 0.3320 & -1.4600 \end{pmatrix}.$$

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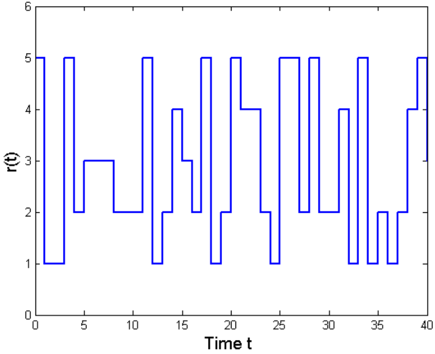

FIGURE: Markov jump $r(t)$ with initial condition $r(0) = 5$

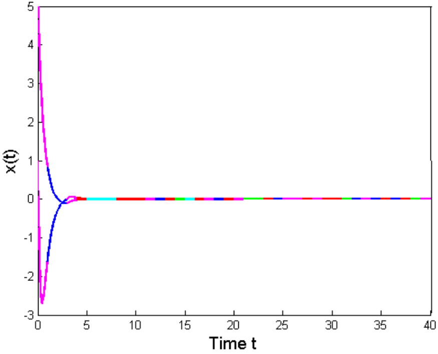

FIGURE: Trajectory solution of System (12) with initial condition $x(0) = [1, 5]^T$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets. $\mathcal{M}^2\left((-\infty, T]; \mathbb{R}^n\right)$ denotes the family of all $\mathcal{F}_t$-measurable $\mathbb{R}^n$ valued processes $x(t)$, $t \in (-\infty, T]$ such that $E\left(\int_{-\infty}^{T} |x(t)|^2 dt\right) < \infty$.

Let $C^\mu = \{\varphi \in C(-\infty; 0]; \mathbb{R}^n : \lim_{\theta\to-\infty} e^{\mu\theta}\varphi(\theta)$ exists in $\mathbb{R}^n\}$ denote the family of continuous functions φ defined on $(-\infty, 0]$ with norm $|\varphi|_\mu = \sup_{\theta\leq 0} e^{\mu\theta}|\varphi(\theta)|$.

1.
2.
3.
4. **Existence and Uniqueness of Solutions of Stochastic Differential Equations with infinite delay**

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets. $\mathcal{M}^2\big((-\infty, T]; \mathbb{R}^n\big)$ denotes the family of all $\mathcal{F}_t$-measurable $\mathbb{R}^n$ valued processes $x(t)$, $t \in (-\infty, T]$ such that $E\Big(\int_{-\infty}^{T} |x(t)|^2 dt\Big) < \infty$.

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Consider the n-dimensional stochastic functional differential equation

$$dx(t) = f(x_t, t)dt + g(x_t, t)dW(t), \quad t_0 \leq t \leq T, \tag{19}$$

where $x_t : (-\infty, 0] \longrightarrow \mathbb{R}^n; \theta \longmapsto x_t(\theta) = x(t+\theta); -\infty < \theta \leq 0$ can be regarded as a C^μ-value stochastic process

$f : C^\mu \times [t_0, T] \to \mathbb{R}^n$ and $g : C^\mu \times [t_0, T] \to \mathbb{R}^{n\times m}$ are Borel measurable.
Assume that $W(t)$ is an m-dimensional Brownian motion which is defined on $(\Omega, \mathcal{F}, P)$.
The initial data of the stochastic process is defined on $(-\infty, t_0]$, with $x_{t_0} = \xi = \{\xi(\theta) : -\infty < \theta \leq 0\}$ $\mathcal{F}_{t_0}$-measurable and $\xi \in \mathcal{M}^2(C^\mu)$.

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Definition 4.1

The $\mathbb{R}^n$-value stochastic process $x(t)$ defined on $-\infty < t \leq T$ is called a solution of (19) with initial data x_{t_0}, if $x(t)$ has the following properties :

- *$x(t)$ is continuous and $\{x(t)\}_{t_0 \leq t \leq T}$ is $\mathcal{F}_t$-adapted,*
- *$\{f(x_t, t)\} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^n)$ and $\{g(x_t, t)\} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{n\times m})$,*
- *$x_{t_0} = \xi$, for each $t_0 \leq t \leq T$,*

$$x(t) = \xi(0) + \int_{t_0}^{t} f(x_s, s)ds + \int_{t_0}^{t} g(x_s, s)dW(s) \text{ almost surely (a.s.)}.$$

$x(t)$ is called a unique solution, if any other solution $\bar{x}(t)$ is distinguishable with $x(t)$, that is

$$P\{x(t) = \bar{x}(t), \text{ for any } 0 \leq t \leq T\} = 1. \quad (20)$$

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Lemma 4.1

If $p \geq 2, g \in \mathcal{M}^2\left([t_0, T]; \mathbb{R}^{n\times m}\right)$ *such that* $E\int_{t_0}^{T} |g(s)|^p ds < \infty$*, then*

$$E\Big|\int_{t_0}^{T} g(s)dW(s)\Big|^p \leq \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_{t_0}^{T} |g(s)|^p ds.$$

Now, we establish existence and uniqueness of solutions for (19) with initial data x_{t_0}.

Theorem 4.2

Assume that there exist two positive numbers K and $\bar{K}$ such that
(i) For any $\varphi, \psi \in C^\mu$ and $t \in [t_0, T]$, it follows that

$$|f(\varphi,t) - f(\psi,t)|^2 \vee |g(\varphi,t) - g(\psi,t)|^2 \leq \bar{K}|\varphi - \psi|_\mu^2 \tag{21}$$

(ii) For any $t \in [t_0, T]$, it follows that $f(0,t), g(0,t) \in L^2(C^\mu)$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K. \tag{22}$$

Then, System (19) with initial data $x_{t_0} = \xi \in \mathcal{M}^2((-\infty,0];\mathbb{R}^n)$, has a unique solution $x(t)$. Moreover, $x(t) \in \mathcal{M}^2\big((-\infty,T];\mathbb{R}^n\big)$.

To show this theorem, we need the following lemma

Lemma 4.3

Let (21) and (22) hold. If $x(t)$ is the solution of (19) with initial data $x_{t_0} = \xi$, then

$$E\Big(\sup_{t_0 \leq t \leq T} |x(t)|^2 \Big) \leq C e^{6\bar{K}(T-t_0+1)(T-t_0)} \tag{23}$$

where $C = 3E|\xi|_\mu^2 + 6K(T-t_0+1)(T-t_0) + 6\bar{K}(T-t_0+1)(T-t_0)E|\xi|_\mu^2$.
Moreover, if $\xi \in \mathcal{M}^2((-\infty,0];\mathbb{R}^n)$, then $x(t) \in \mathcal{M}^2((-\infty,T];\mathbb{R}^n)$.

Sketch of proof

- For each number $q \geq 1$, define the stoping time
$$\tau_q = T \wedge \inf\{t \in [t_0, T] : |x_t|_\mu \geq q\}. \quad (24)$$
Obviously, as $q \to \infty, \tau_q \nearrow T$ a.s.
- Let $x^q(t) = x(t \wedge \tau_q), t \in [t_0, T]$, then $x^q(t)$ satisfy the following equation
$$x^q(t) = \xi(0) + \int_{t_0}^{t} f(x_s^q, s) I_{[t_0, \tau_q]}(s) ds + \int_{t_0}^{t} g(x_s^q, s) I_{[t_0, \tau_q]}(s) dW(s) \quad (25)$$
- By using the Hölder inequality, Lemma 4.1, (21) and (22), we show that
$$E|x^q(t)|^2 \leq 3E|\xi|_\mu^2 + 6K(t - t_0 + 1)(t - t_0) + 6\bar{K}(t - t_0 + 1)E \int_{t_0}^{t} |x_s^q|_\mu^2 ds.$$

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Sketch of proof

- Using some properties of the norm, we get

$$E\Big(\sup_{t_0\leq s\leq T}|x^q(s)|^2\Big)\leq C+6\bar{K}(T-t_0+1)\int_{t_0}^{T}E\Big(\sup_{t_0\leq r\leq T}|x^q(r)|^2\Big)dr$$

where $C=3E|\xi|_\mu^2+6K(T-t_0+1)(T-t_0)+6\bar{K}(T-t_0+1)(T-t_0)E|\xi|_\mu^2$.

- By the Gronwall inequality, we infer

$$E\Big(\sup_{t_0\leq s\leq T}|x(s\wedge\tau_q)|^2\Big)\leq Ce^{6\bar{K}(T-t_0+1)(T-t_0)}.$$

- Letting $q\longrightarrow\infty$, that implies the following inequality

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$$E\Big(\sup_{t_0\le s\le T}|x(s)|^2\Big)\le Ce^{6\bar{K}(T-t_0+1)(T-t_0)}.$$

Next, to prove the second part or the lemma, suppose that $\xi \in \mathcal{M}^2((-\infty,0];\mathbb{R}^n)$. Then

$$\begin{aligned}
E\Big(\sup_{-\infty\leq t\leq T}|x(t)|^2\Big) &= E\Big(\sup_{-\infty\leq t\leq t_0}|x(t)|^2\Big) + E\Big(\sup_{t_0\leq t\leq T}|x(t)|^2\Big)\\
&\leq E\Big(\sup_{-\infty\leq t\leq t_0}|x(t)|^2\Big) + Ce^{6\bar{K}(T-t_0+1)(T-t_0)}\\
&\leq E\Big(\sup_{-\infty\leq t-t_0\leq 0}|x(t-t_0+t_0)|^2\Big)\\
&+ Ce^{6\bar{K}(T-t_0+1)(T-t_0)}\\
&\leq E\Big(\sup_{-\infty\leq s\leq 0}|x_{t_0}(s)|^2\Big) + Ce^{6\bar{K}(T-t_0+1)(T-t_0)}\\
&\leq E|\xi|^2 + Ce^{6\bar{K}(T-t_0+1)(T-t_0)}\\
&< \infty.
\end{aligned}$$

Proof of theorem 4.2

We begin by check uniqueness of solution.

- Let $x(t)$ and $\bar{x}(t)$ be two solutions of (19), by lemma 4.3 $x(t)$ and $\bar{x}(t) \in \mathcal{M}^2\big((-\infty, T]; \mathbb{R}^n\big)$. Note that

$$x(t) - \bar{x}(t) = \int_{t_0}^{t} \big[f(x_s, s) - f(\bar{x}_s, s)\big]ds + \int_{t_0}^{t} \big[g(x_s, s) - g(\bar{x}_s, s)\big]dW(s)$$

- By Hölder inequality, Lemma 4.1, (21), (22) and properties of the norm, we show that

$$E\Big(\sup_{t_0 \le s \le t} \big|x(s) - \bar{x}(s)\big|^2\Big) \le 2\bar{K}(t - t_0 + 1)\int_{t_0}^{t} E\Big(\sup_{t_0 \le s \le t} \big|x(s) - \bar{x}(s)\big|^2\Big)ds.$$

- Applying the Gronwall inequality to yield

$$E\big(|x(t) - \bar{x}(t)|^2\big) = 0, \quad t_0 \le t \le T$$

That is $x(t) = \bar{x}(t)$ a.s. for $t_0 \le t \le T$. Therefore, for all $-\infty < t \le T, x(t) = \bar{x}(t)$ a.s.

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- Next, to check the existence, define $x^0_{t_0} = \xi$ and $x^0(t) = \xi(0)$ for $t_0 \leq t \leq T$. Let $x^k_{t_0} = \xi, k = 1, 2, ...,$ and define Picard sequence

$$x^k(t) = \xi(0) + \int_{t_0}^{t} f(x_s^{k-1}, s)ds + \int_{t_0}^{t} g(x_s^{k-1}, s)dW(s), \quad t_0 \leq t \leq T \qquad (26)$$

- Obviously $x^0(t) \in \mathcal{M}^2((-\infty, T]; \mathbb{R}^n)$. By induction, we can see that $x^k(t) \in \mathcal{M}^2((-\infty, T]; \mathbb{R}^n)$.
- By Hölder inequality, Lemma 4.1, (21) and (22)

$$E|x^k(t)|^2 \leq C_1 + C_2 + 6\bar{K}(t - t_0 + 1)E\int_{t_0}^{t} |x^{k-1}(s)|^2 ds$$

where $C_1 = 3E|\xi|^2_\mu + 6K(t - t_0 + 1)(t - t_0)$
and $C_2 = 6\bar{K}(t - t_0 + 1)(t - t_0)E|\xi|^2_\mu$.

- Next, to check the existence, define $x^0_{t_0} = \xi$ and $x^0(t) = \xi(0)$ for $t_0 \leq t \leq T$. Let $x^k_{t_0} = \xi, k = 1, 2, ...,$ and define Picard sequence

$$x^k(t) = \xi(0) + \int_{t_0}^{t} f(x_s^{k-1}, s)ds + \int_{t_0}^{t} g(x_s^{k-1}, s)dW(s), \quad t_0 \leq t \leq T \tag{26}$$

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- By using Gronwall inequality

$$E|x^k(t)|^2 \leq C_3 e^{6\bar{K}(T-t_0+1)(T-t_0)} \quad t_0 \leq t \leq T, k \geq 1.$$

where $C_3 = C_1 + 2C_2$.

- From the Hölder inequality, Lemma 4.1, (21) and (22), as in a similar earlier inequality, one then has

$$E\Big(\sup_{t_0 \leq s \leq t} |x^1(s) - x^0(s)|^2\Big) \leq 4(T - t_0 + 1)(T - t_0)(K + \bar{K}E|\xi|_\mu^2) := R.$$

- By similar arguments as above, we also have

$$E\Big(\sup_{t_0 \leq s \leq t} |x^2(s) - x^1(s)|^2\Big) \leq 2R\bar{K}(T - t_0 + 1)(T - t_0) = RM(T - t_0),$$

where $M = 2\bar{K}(T - t_0 + 1)$.

- Continuing this process to find that

$$E\Big(\sup_{t_0 \leq s \leq t} |x^{k+1}(s) - x^k(s)|^2\Big) \leq \frac{R[M(T - t_0)]^k}{k!}, \quad t_0 \leq t \leq T$$

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- By Alembert's rule, Chebyshev inequality and Borel-Cantelli's Lemma, we show that $\{x^k(t)\}$ is also a Cauchy sequence in L^2. Hence, $\{x^k(t)\}$ converges uniformly and let $x(t)$ be its limit for any $t \in (-\infty, T]$
- Noting that the sequence $\{x^k(t)\} \to x(t)$ means that for any given $\varepsilon > 0$ there exist k_0 such that when $k \geq k_0$, for any $t \in (-\infty, T]$, one has

$$E|x^k(t) - x(t)|^2 < \varepsilon, \quad \text{and} \int_{t_0}^{T} E|x^k(t) - x(t)|^2 dt < (T - t_0)\varepsilon.$$

- Which implies that

$$\int_{t_0}^{t} f(x_s^k, s)ds \to \int_{t_0}^{t} f(x_s, s)ds \text{ and } \int_{t_0}^{t} g(x_s^k, s)dW(s) \to \int_{t_0}^{t} g(x_s, s)dW(s) \text{ in } L^2.$$

- Taking limits on both sides of (26), we obtain

$$x(t) = \xi(0) + \int_{t_0}^{t} f(x_s, s)ds + \int_{t_0}^{t} g(x_s, s)dW(s) \quad t_0 \leq t \leq T.$$

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5. **Stability Analysis of Stochastic Neural Network with Infinite Delay and Markovian jump**

We define $C^{\mu}_{\alpha} \triangleq \{\phi \in C^{\mu}; |\phi|_{\mu} < \alpha\}$.
For any $M > 0$, define two random variables τ^{y}_{M} and τ^{M}_{y} as follows :

$$\tau^{y}_{M} = \inf\{t \geq t_0 : |y(t)| \geq M, |\xi|_{\mu} < M, a.s.\}$$

$$\tau^{M}_{y} = \inf\{t \geq t_0 : |y(t)| \leq M, |\xi|_{\mu} > M, a.s.\},$$

where $y : [0, +\infty) \times \Omega \longrightarrow \mathbb{R}$ is a continuous stochastic process.

The general neural networks (NNs) with infinite delay can be described by a Volterra integro-differential equation :

$$\dot{u}(t) = -Du(t) + Ag(u(t)) + \int_{-\infty}^{t} CK^T(t-s)g(u(s))ds + J, \tag{27}$$

where $u(t) = (u_1(t), u_2(t), ..., u_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons, $D = diag(d_1, d_2, ..., d_n) \gg 0$ is the firing rate of the neurons, $A = (a_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$ are connection weight matrices, $g(u) = (g_1(u_1), g_2(u_2), ..., g_n(u_n)^T$ is the neuron activation function vector, $K = (K_{ij})_{n\times n}$ such that $K_{ij} : [0, +\infty) \longrightarrow [0, +\infty)$ $(i, j = 1, 2, ..., n)$ are piecewise continuous on $[0, +\infty)$ satisfying

$$\int_{0}^{+\infty} K_{ij}(s)e^{\mu s}ds = \bar{K}. \qquad i, j = 1, 2, ..., n. \tag{28}$$

where $\bar{K}$ is a positive constant depending on μ.
and $J = (J_1, J_2, ..., J_n)^T$ is the constant external input vector

By making a transformation $x(t) = u(t) - u^*$, System (27) has a unique equilibrium point, and it can be rewritten as

$$\dot{x}(t) = -Dx(t) + AF(x(t)) + \int_{-\infty}^{t} CK^T(t-s)F(x(s))ds, \tag{29}$$

where $F(x(t)) = \big(g_1(x_1(t)+u_1^*), ..., g_n(x_n(t)+u_n^*)\big)^T \triangleq \big(f_1(x_1(t)), ..., f_n(x_n(t))\big)^T$.
Consider System (29) disturbed by white noise and Markovian switching, which, naturally, called stochastic neural networks with infinite delay and Markovian switching as follows :

$$dx(t) = \Big[-Dx(t) + A(r(t))F(x(t)) + \int_{-\infty}^{t} C(r(t))K^T(t-s)F(x(s))ds\Big]dt + B(r(t))Q(x(t))dW(t), \tag{30}$$

where $B(r(t)) = \big(b_{ij}(r(t))\big)_{n\times n}$ and $Q(x) = \big(q_1(x_1(t)), q_2(x_2(t)), ..., q_n(x_n(t))\big)^T$ represents the disturbance intensity of white noise satisfying $Q(0) = 0$.
We also assume that Markov chain $r(t)$ is independent of Brownian motion $W(t)$, and it is irreducible.

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For any $(\phi, k) \in C^{\mu} \times \mathcal{M}$, we denote

$$\begin{cases} \mathbb{E}\big(\phi, k\big) = -D\phi(0) + A(k)F(\phi(0)) + \int_{-\infty}^{t} C(k)K^{T}(t-s)F(\phi(s-t))ds, \\ \mathbb{H}\big(\phi, k\big) = B(k)Q(\phi(0)). \end{cases}$$

If $V \in \mathcal{C}^2\big(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{M}; \mathbb{R}_+\big)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{M}$ to $\mathbb{R}$ by

$$\mathcal{L}V(x,t,k) = V_t(x,t,k) + V_x(x,t,k)\mathbb{E}(x_t,k) + \frac{1}{2}Trace\big[\mathbb{H}^T(x_t,k)V_{xx}(x,t,k)\mathbb{H}(x_t,k)\big] + \sum_{\ell=1}^{N}\gamma_{k\ell}V(x,t,\ell), \qquad (31)$$

where

$$V_t(x,t,k) = \frac{\partial V(x,t,k)}{\partial t}, \quad V_x(x,t,k) = \Big(\frac{\partial V(x,t,k)}{\partial x_1}, \ldots, \frac{\partial V(x,t,k)}{\partial x_n}\Big)$$

and

$$V_{xx}(x,t,k) = \Big(\frac{\partial^2 V(x,t,k)}{\partial x_i \partial x_j}\Big)_{n\times n}$$

For any $(\phi, k) \in C^{\mu} \times \mathcal{M}$, we denote

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Assumption 5.1

For each $j \in \{1, 2, ..., n\}$, functions $g_j : \mathbb{R} \longrightarrow \mathbb{R}$ and $q_j : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy global Lipschitz conditions

$$|g_j(x) - g_j(y)| \vee |q_j(x) - q_j(y)| \leq L_j|x - y|, \quad \text{for} \quad x, y \in \mathbb{R}, \tag{32}$$

that is,

$$|F(x)| \vee |Q(x)| \leq L|x| \tag{33}$$

where $L = \max\{L_1, L_2, ..., L_n\}$. In addition, the initial data $x_{t_0} = \xi$ satisfies $|\xi| := \sup_{\theta \leq 0} |\xi(\theta)| < \infty$.

Theorem 5.1

Suppose that Assumption 5.1 holds. Then System (30) has a unique global solution on $(-\infty, \infty)$ *with initial data* $\xi \in C^{\mu}$ *and* $r(t_0) = r_0$.

Sketch of proof

- By definition of the right continuous Markov jump $r(.)$, there is a sequence $\{\tau_k\}_{k\geq 0}$ of stoping times such that $r(.)$ is a random constant on every interval $[\tau_k, \tau_{k+1})$, that is $r(t) = r(\tau_k)$ on $\tau_k \leq t < \tau_{k+1}$, for any $k \geq 0$.
- We consider first System (30) for $t \in [\tau_0, \tau_1)$, we can rewrite it as

$$dx(t) = \mathbb{E}(x_t, r_0)dt + \mathbb{H}(x_t, r_0)dW(t) \quad (34)$$

- By calculation, we get

$$|\mathbb{E}(\xi, r_0) - \mathbb{E}(\zeta, r_0)| \leq \left(|D| + L|A(r_0)| + n^2 L|C(r_0)|\bar{K}\right)|\xi - \zeta|_\mu$$

$$\text{and} \qquad |\mathbb{H}(\xi, r_0) - \mathbb{H}(\zeta, r_0)| \leq L|B(r_0)||\xi - \zeta|_\mu$$

- Then, by Theorem 4.2 System (30) with initial condition $\xi \in C^\mu$ and $r(t_0) = r_0$ has a unique solution on $[\tau_0, \tau_1)$.
- By the same argument of existence and uniqueness as the first step above, System (30) with initial condition $\xi \in C^\mu$ and $r(0) = r_0$ has a unique solution on $(-\infty, \infty)$.

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- By calculation, we get

$$|\mathbb{E}(\xi, r_0) - \mathbb{E}(\zeta, r_0)| \leq \Big(|D| + L|A(r_0)| + n^2 L|C(r_0)|\bar{K}\Big)|\xi - \zeta|_\mu$$

and $$|\mathbb{H}(\xi, r_0) - \mathbb{H}(\zeta, r_0)| \leq L|B(r_0)||\xi - \zeta|_\mu$$

- Then, by Theorem 4.2 System (30) with initial condition $\xi \in C^\mu$ and $r(t_0) = r_0$ has a unique solution on $[\tau_0, \tau_1)$.
- By the same argument of existence and uniqueness as the first step above, System (30) with initial condition $\xi \in C^\mu$ and $r(0) = r_0$ has a unique solution on $(-\infty, \infty)$.

Sketch of proof

- By definition of the right continuous Markov jump $r(.)$, there is a sequence $\{\tau_k\}_{k\geq 0}$ of stoping times such that $r(.)$ is a random constant on every interval $[\tau_k, \tau_{k+1})$, that is $r(t) = r(\tau_k)$ on $\tau_k \leq t < \tau_{k+1}$, for any $k \geq 0$.
- We consider first System (30) for $t \in [\tau_0, \tau_1)$, we can rewrite it as

$$dx(t) = \mathbb{E}(x_t, r_0)dt + \mathbb{H}(x_t, r_0)dW(t) \tag{34}$$

- By calculation, we get

$$|\mathbb{E}(\xi, r_0) - \mathbb{E}(\zeta, r_0)| \leq \Big(|D| + L|A(r_0)| + n^2 L|C(r_0)|\bar{K}\Big)|\xi - \zeta|_\mu$$

$$\text{and} \qquad |\mathbb{H}(\xi, r_0) - \mathbb{H}(\zeta, r_0)| \leq L|B(r_0)||\xi - \zeta|_\mu$$

- Then, by Theorem 4.2 System (30) with initial condition $\xi \in C^\mu$ and $r(t_0) = r_0$ has a unique solution on $[\tau_0, \tau_1)$.
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Definition 5.1

The solution of System (30) with initial data $x_{t_0} = \xi$ *is said to be stochastically stable if for every pair* $\varepsilon \in (0,1)$ *and* $\alpha > 0$, *there exists a* $\delta = \delta(\varepsilon, \alpha) > 0$ *such that*

$$P\Big\{|x(t,t_0,\xi)| < \alpha, t \geq t_0\Big\} \geq 1 - \varepsilon,$$

whenever $(\xi, k) \in C_\delta^\mu \times \mathcal{M}$.

Definition 5.2

The solution of System (30) with initial data $x_{t_0} = \xi$ *is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every* $\varepsilon \in (0,1)$, *there exist* $\delta_0 = \delta_0(\varepsilon) > 0$ *such that*

$$P\Big\{\lim_{t\to\infty} x(t,t_0,\xi) = 0\Big\} \geq 1 - \varepsilon,$$

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The solution of System (30) with initial data $x_{t_0} = \xi$ is said to be stochastically stable if for every pair $\varepsilon \in (0, 1)$ and $\alpha > 0$, there exists a $\delta = \delta(\varepsilon, \alpha) > 0$ such that

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whenever $(\xi, k) \in C^{\mu}_{\delta_0} \times \mathcal{M}$.

Definition 5.3

The solution of System (30) with initial data $x_{t_0} = \xi$ is said to be globally stochastically asymptotically stable if it is stochastically stable and, moreover, for any $(\xi, k) \in C^{\mu} \times \mathcal{M}$,

$$P\{\lim_{t\to\infty} x(t, t_0, \xi) = 0\} = 1.$$

Let $\mathcal{A} := -diag(2\beta_1, 2\beta_2, ..., 2\beta_N) - \mathcal{Q}$ where $d = \min\{d_1, d_2, ..., d_n\}$. and

$$\beta_k := -d + L|A(k)| + \frac{1}{2}L^2|B(k)|^2 + n^2\bar{K}L|C(k)|, \quad k \in \mathcal{M}.$$

Definition 5.3

The solution of System (30) with initial data $x_{t_0} = \xi$ is said to be globally stochastically asymptotically stable if it is stochastically stable and, moreover, for any $(\xi, k) \in C^{\mu} \times \mathcal{M}$,

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Assumption 5.2

There is a $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)^T \geq 0$ *in* $\mathbb{R}^N$ *such that* $P = \mathcal{A}\lambda \geq 0$.

Theorem 5.2

Suppose that Assumptions 5.1 and 5.2 hold. Then the trivial solution to System (30) is stochastically stable.

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Suppose that Assumptions 5.1 and 5.2 hold. Then the trivial solution to System (30) is stochastically stable.

Sketch of proof

- For any $\varepsilon \in (0.1)$, and $\alpha > 0$, we choose a sufficiently small $\delta(\varepsilon, \alpha)$, such that for any $\xi \in C^{\mu}_{\delta(\varepsilon,\alpha)}$,
$$\lambda_k|\xi|^2_\mu + 2n^2\bar{K}L|\xi|_\mu < \lambda_k\varepsilon\alpha^2 \text{ for any } k \in \mathcal{M}$$
For $t \geq t_0, \quad k = 1, 2, ..., N$, let
$$V(x,t,k) = \frac{1}{2}\lambda_k|x|^2 + \int_t^{+\infty}\sum_{i=1}^n\sum_{j=1}^n K_{ij}(s-t)|f_j(x_j(2t-s))|ds \quad (35)$$

- From Assumptions 5.1, 5.2, using the fact that $x(t) = x(t+0) = x_t(0)$ and the transformation $v = t - s$, we show that
$$\mathcal{L}V(x(t),t,k) \leq -\frac{1}{2}p_k|x_t|^2_\mu.$$

- Other by Dynkin formula, Assumption 5.1 and Eq. (28) we infer
$$\frac{\lambda_k}{2}\alpha^2 P(\tau_x^\alpha < t) \leq V(x(t\wedge\tau_x^\alpha), t\wedge\tau_x^\alpha, k) \leq EV(x(t_0), t_0, k)$$
$$\leq \frac{\lambda_k}{2}E|\xi|^2_\mu + n^2\bar{K}LE|\xi|_\mu < \frac{\lambda_k}{2}\varepsilon\alpha^2$$

- Letting $t \longrightarrow \infty$ we have $P\{\tau_x^\alpha < \infty\} < \varepsilon$, which is equivalent to
$$P\{|x(t,t_0,\xi)| \leq \alpha, t \geq t_0\} \geq 1 - \varepsilon. \quad (36)$$

Sketch of proof

- For any $\varepsilon \in (0.1)$, and $\alpha > 0$, we choose a sufficiently small $\delta(\varepsilon, \alpha)$, such that for any $\xi \in C^{\mu}_{\delta(\varepsilon,\alpha)}$,
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Sketch of proof

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Assumption 5.3

If $\mathcal{A}$ is a nonsingular M-matrix, there is a $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)^T \gg 0$ in $\mathbb{R}^N$ such that $P = \mathcal{A}\lambda \gg 0$.

Theorem 5.3

Suppose that Assumptions 5.1 and 5.3 hold. Then the solution to System (30) is stochastically asymptotically stable.

Sketch of proof

Lemma 5.4

Suppose Assumptions 5.1 and 5.2 hold. Then for any $\varepsilon \in (0,1)$ and $\alpha > 0$, there exists a $R(\varepsilon, \alpha) > 0$, such that for any $t_0 \geq 0$ and $\xi \in C^{\mu}_{\alpha}$ a.s.,

$$P\{|x(t,t_0,\xi)| \leq R, t \geq t_0\} \geq 1 - \varepsilon.$$

- From Theorem 5.2, we can easily see that the trivial solution to System (30) is stochastically stable, that is, for any $\delta_1 > 0$ and $\varepsilon \in (0,1)$, there exists a $\delta(\varepsilon, \delta_1) > 0$ such that for any $\xi \in C^{\mu}_{\delta(\varepsilon,\delta_1)}$,

$$P(A) \geq 1 - \varepsilon,$$

in which $A \triangleq \{\omega : |x(t,t_0,\xi)| < \delta, t \geq t_0\}$.

Sketch of proof

Lemma 5.4

Suppose Assumptions 5.1 and 5.2 hold. Then for any $\varepsilon \in (0,1)$ and $\alpha > 0$, there exists a $R(\varepsilon, \alpha) > 0$, such that for any $t_0 \geq 0$ and $\xi \in C^{\mu}_{\alpha}$ a.s.,

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- From Theorem 5.2, we can easily see that the trivial solution to System (30) is stochastically stable, that is, for any $\delta_1 > 0$ and $\varepsilon \in (0,1)$, there exists a $\delta(\varepsilon, \delta_1) > 0$ such that for any $\xi \in C^{\mu}_{\delta(\varepsilon,\delta_1)}$,

$$P(A) \geq 1 - \varepsilon,$$

in which $A \triangleq \{\omega : |x(t,t_0,\xi)| < \delta, t \geq t_0\}$.

Sketch of proof

- From Lemma 5.4 it follows immediately that for δ_1 and any $\varepsilon_1 \in (0,1)$, there exists a $H(\varepsilon_1, \delta_1)$ sufficiently large such that

$$P\{|x(t,\theta^*,\xi_{\theta^*})| \leq H, t \geq \theta^*\} \geq 1 - \frac{\varepsilon_1}{4}, \quad \text{and } |x_{\theta^*}|_\mu < H, \forall \theta^* \leq t. \qquad (37)$$

- Next, we show that if there exists a $k > 0$, such that

$$P\{\omega \in A : |x(\tau_k, t_0, \xi)| = 0, t \geq k\} = P(A) \geq 1 - \varepsilon,$$

then the trivial solution to System (30) is stochastically asymptotically stable

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Theorem 5.5

Suppose that Assumptions 5.1 and 5.3 hold. Then the solution to System (30) is globally stochastically asymptotically stable.

Sketch of proof

- By Theorem 5.2, the solution of System (30) is stochastically stable. So we only need to show that for any $\xi \in C^{\mu}$,

$$P\{\lim_{t\to\infty} x(t, t_0, \xi) = 0\} = 1.$$

- Fix any $\varepsilon \in (0, 1)$ and $\xi \in C^{\mu}$. Let

$$V(x, t, k) = \frac{\lambda_k}{2}\sum_{i=1}^{n} x_i^2 + \int_t^{+\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij}(s-t)|f_j(x_j(2t-s))|ds.$$

- Let H be sufficiently large such that

$$\inf_{\omega\in\Omega, |x|>H, t\geq t_0} V(x, t, k) \geq \frac{V(\xi(0), t_0, k)}{\varepsilon}.$$

- By the generalized Itô formula, we infer

$$P\{\tau_x^H < t\} \leq \varepsilon\frac{V(x(t\wedge\tau_x^H), t\wedge\tau_x^H, k)}{V(\xi(0), t_0, k)} < \varepsilon.$$

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- By the generalized Itô formula, we infer

$$P\{\tau_x^H < t\} \leq \varepsilon \frac{V(x(t\wedge\tau_x^H), t\wedge\tau_x^H, k)}{V(\xi(0), t_0, k)} < \varepsilon.$$

- Let $t \longrightarrow \infty$. Then

$$P\{\tau_x^H < \infty\} < \varepsilon$$

namely,

$$P\{\sup_{t \geq t_0} |x(t, t_0, \xi)| \leq H\} \geq 1 - \varepsilon.$$

- From here, following the proof of Theorem 5.3 we can easily find that

$$P\{\lim_{t \to \infty} x(t, t_0, \xi) = 0\} \geq 1 - \varepsilon.$$

- Since ε is arbitrary, the trivial solution to system (30) is globally stochastically asymptotically stable.

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- Since ε is arbitrary, the trivial solution to system (30) is globally stochastically asymptotically stable.

Example 5.1

- *Let* $r(t)$ *be a right-continuous Markovian process taking values in* $\mathcal{M} = \{1, 2, 3\}$ *with generator*

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 2 & -5 \end{pmatrix}$$

- *Consider a two-dimensional System (30) with the following specification*

$$D = \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}, A(1) = \begin{pmatrix} 2 & 1 \\ 1 & 1.5 \end{pmatrix}, B(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C(1) = \begin{pmatrix} 0.5 & 0 \\ 0 & \sqrt{2}, \end{pmatrix},$$

$$A(2) = \begin{pmatrix} 2 & 0.5 \\ 0.3 & 0.8 \end{pmatrix}, B(2) = \begin{pmatrix} \sqrt{0.2} & 0 \\ 0 & \sqrt{0.2} \end{pmatrix}, C(2) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix},$$

$$A(3) = \begin{pmatrix} 2 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}, B(3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C(3) = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.5, \end{pmatrix}.$$

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- We rewrite System (30) in detailed form

$$\begin{cases} dx(t) &= \Big[-15x(t) + a_{11}(r(t))h(x(t)) + a_{12}(r(t))h(y(t)) \\ & \quad + \int_{-\infty}^{t} c_{11}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \Big] dt + b_{11}(r(t))q_1(x(t))dW(t), \\ dy(t) &= \Big[-15x(t) + a_{21}(r(t))h(x(t)) + a_{22}(r(t))h(y(t)) \\ & \quad + \int_{-\infty}^{t} c_{22}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \Big] dt + b_{22}(r(t))q_2(x(t))dW(t), \end{cases} \tag{38}$$

where $q_1(x) = q_2(x) = \sin x$ satisfies global Lipschitz condition with Lipschitz constant $L = 1, h(x) = \sin x$, this means that Assumption 5.1 is verified.

- To validate Assumption 5.3, let $\mu = 0.4$, then

$$\mathcal{A} = \begin{pmatrix} +4.2554 & -1.000 & -1.0000 \\ -2.0000 & +6.0428 & -2.0000 \\ -3.0000 & -2.0000 & 21.0421 \end{pmatrix}.$$

- Hence, it is desired that $\mathcal{A}$ is a nonsingular M-matrix.
  By Theorem 5.5, System (30) is globally stochastically asymptotically stable.

- We rewrite System (30) in detailed form

$$\begin{cases} dx(t) & = \Big[-15x(t) + a_{11}(r(t))h(x(t)) + a_{12}(r(t))h(y(t)) \\ & + \int_{-\infty}^{t} c_{11}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \Big] dt + b_{11}(r(t))q_1(x(t))dW(t), \\ dy(t) & = \Big[-15x(t) + a_{21}(r(t))h(x(t)) + a_{22}(r(t))h(y(t)) \\ & + \int_{-\infty}^{t} c_{22}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \Big] dt + b_{22}(r(t))q_2(x(t))dW(t), \end{cases} \tag{38}$$

where $q_1(x) = q_2(x) = \sin x$ satisfies global Lipschitz condition with Lipschitz constant $L = 1, h(x) = \sin x$, this means that Assumption 5.1 is verified.

- To validate Assumption 5.3, let $\mu = 0.4$, then

$$\mathcal{A} = \begin{pmatrix} +4.2554 & -1.000 & -1.0000 \\ -2.0000 & +6.0428 & -2.0000 \\ -3.0000 & -2.0000 & 21.0421 \end{pmatrix}.$$

- Hence, it is desired that $\mathcal{A}$ is a nonsingular M-matrix.
By Theorem 5.5, System (30) is globally stochastically asymptotically stable.

- We rewrite System (30) in detailed form

$$\begin{cases} dx(t) & = \Big[-15x(t) + a_{11}(r(t))h(x(t)) + a_{12}(r(t))h(y(t)) \\ & \quad + \int_{-\infty}^{t} c_{11}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \Big] dt + b_{11}(r(t))q_1(x(t))dW(t), \\ dy(t) & = \Big[-15x(t) + a_{21}(r(t))h(x(t)) + a_{22}(r(t))h(y(t)) \\ & \quad + \int_{-\infty}^{t} c_{22}(r(t))e^{s-t}(h(x(s)) + h(y(s)))ds \Big] dt + b_{22}(r(t))q_2(x(t))dW(t), \end{cases} \tag{38}$$

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  By Theorem 5.5, System (30) is globally stochastically asymptotically stable.

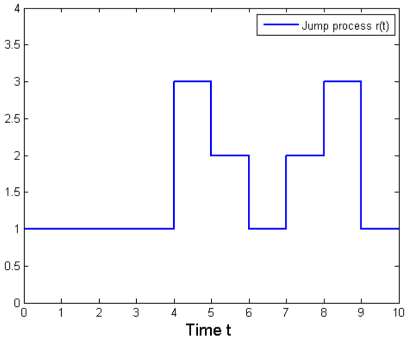

FIGURE: Jump process r(t) with initial condition r(0)=1

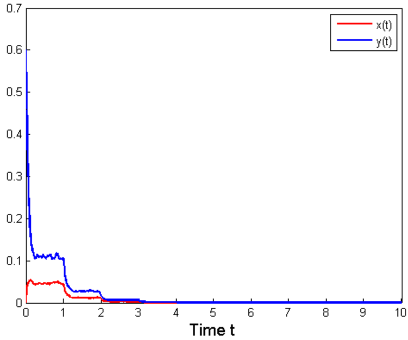

FIGURE: Approximate solution of System (38) with initial conditions $x(0) = \sin^2(0)$, $y(0) = 0.6$

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6 **Conclusion and Perspectives**

Our work concerns stochastic stability analysis of hybrid dynamical systems with Markovian switching, using Lyapunov method, M-matrix theory and stochastic analysis.

As perspectives :

- We intend to collaborate with researchers in electrical engineering for the application of hybrid dynamic systems with Markovian switching, especially for studying processes altered by abrupt variations.
- We project to work on other mathematical aspects of study such as
  - Stochastic optimal control
  - Infinite dimension
  - ...

Thank you for your Attention

Merci pour votre Attention

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- C. Imzegouan, H. Bouzahir, B. Benaid and F. El Guezar, A Note on Exponential Stochastic Stability of Markovian Switching Systems, International Journal of Evolution Equations, Vol 10, Issue 2, (2016), pp. 189-198.
- C. Imzegouan, Stochastic Stability in terms of an Associated Transfer Function Matrix for Some Hybrid Systems with Markovian Switching. Commun. Fac. Sci. Univ. Ankara, Ser. A1, Math. Stat. Volum 67, Number 1, pages 1-0 (2018)
- H. Bouzahir, B. Benaid and C. Imzegouan, Some Stochastic Functional Differential Equations with Infinite Delay : A Result on Existence and Uniqueness of Solutions in a Concrete Fading Memory Space. Chin. J. Math. (N.Y.) 2017.
- B. Benaid, H. Bouzahir, C. Imzegouan and F. El Guezar Stochastic Stability Analysis for Stochastic Neural Networks with Markovian Switching and Infinite Delay in a phase space, (In revision).