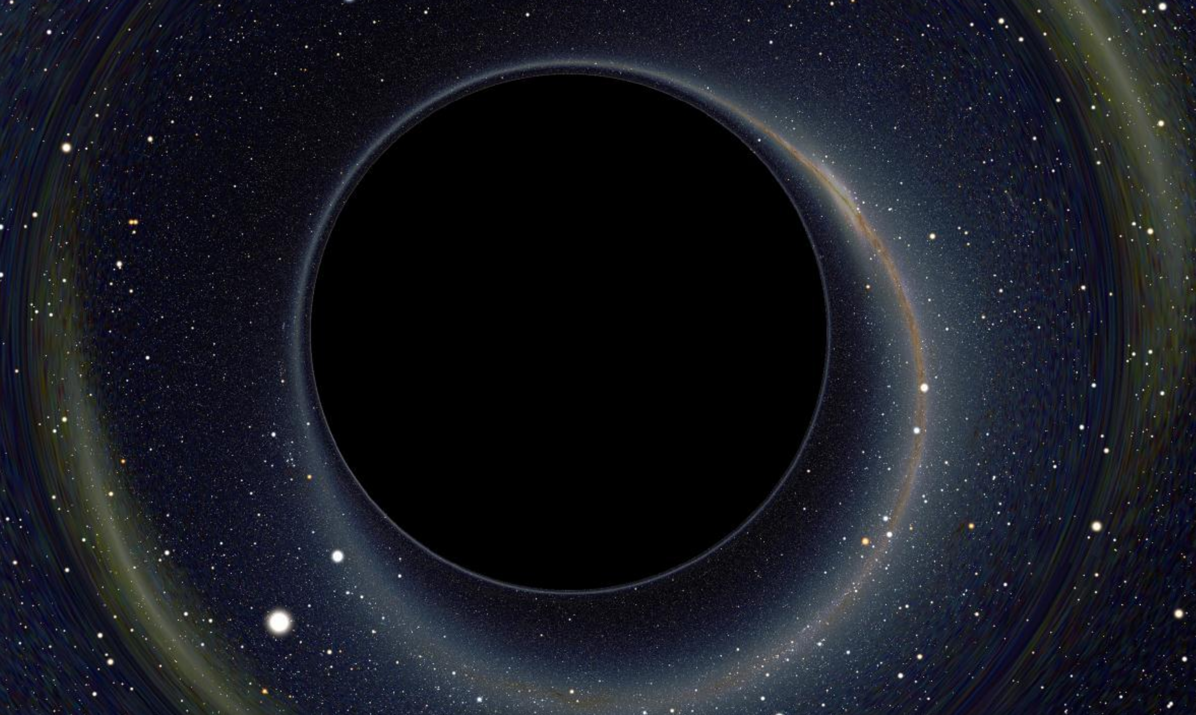
Particle Dynamics Around Black Holes:

An Implication to Gravitational Lensing



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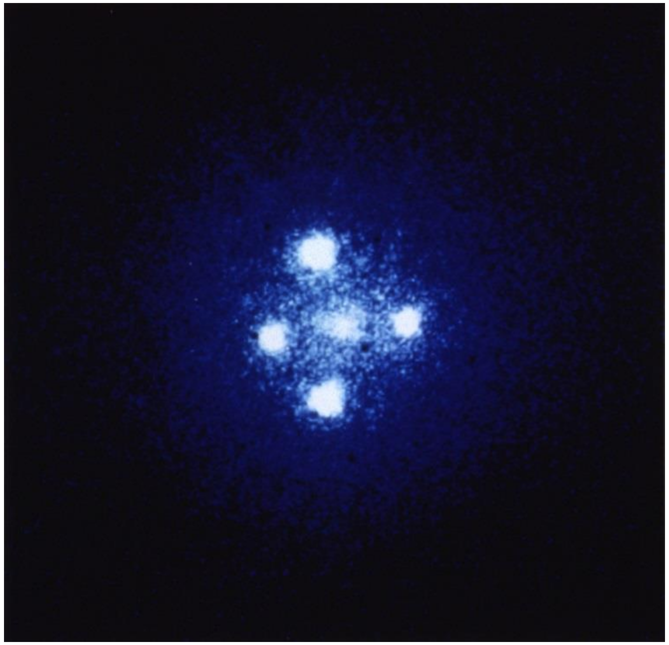
Defining futures



A terrestrial mirage versus a cosmic mirage

Outline

- Gravity as geometry
- Basics of gravitational lensing (GL)
- Black holes and null geodesics
- GL by black holes
- Final remarks

Einstein field equations are used to investigate strong gravitational fields in the Universe:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3$$

(Origin of spacetime curvature is matter !)

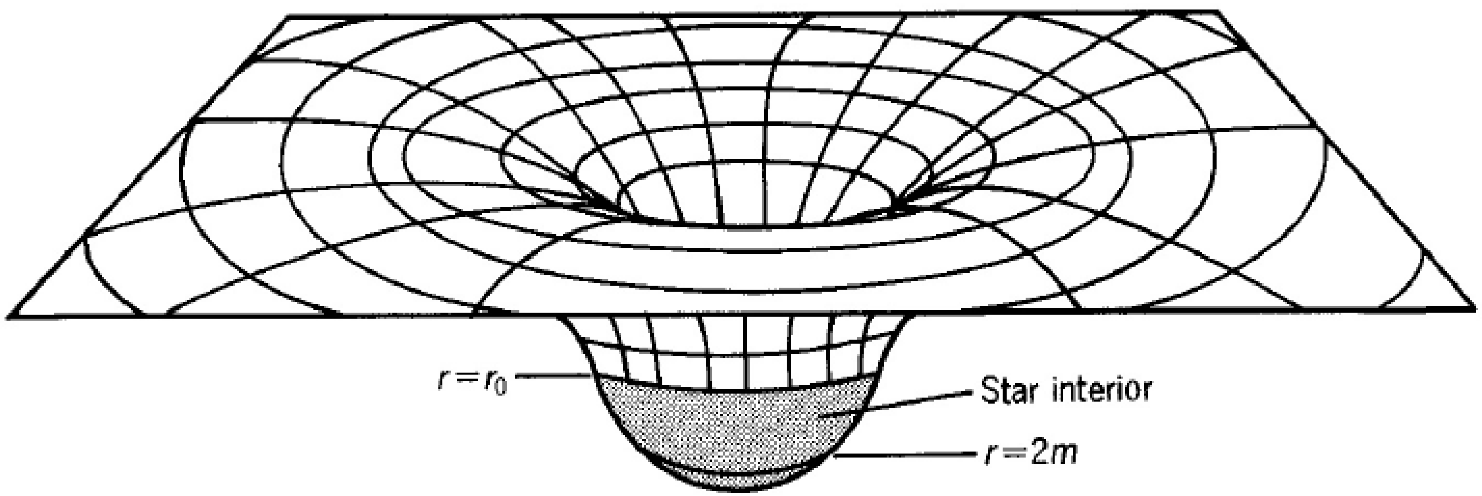
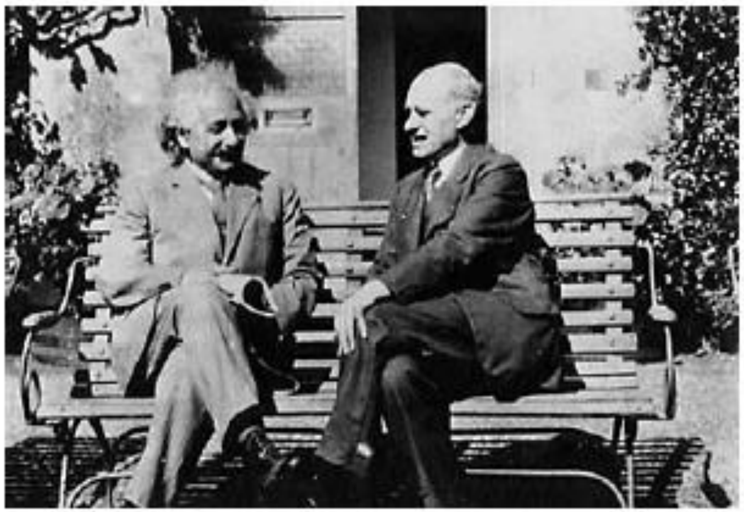


Figure : Gravity should be understood as curvature of spacetime (and NOT as a Force).

Path of Light Bends Near the Sun



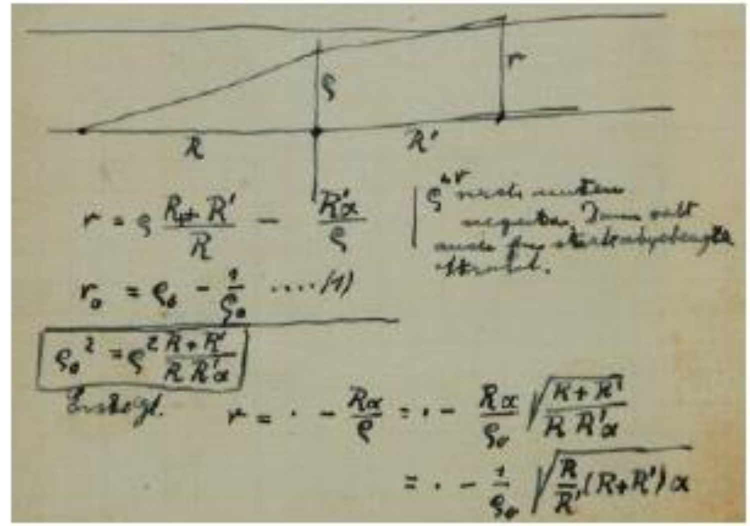


Figure : (*Left*) Albert Einstein predicted in 1913 that light of a distant star passing by the Sun will be deflected due to curvature. Sir Arthur Eddington in 1919 tested the prediction and found it to be consistent with observations. (*Right*) Einstein's hand calculations regarding bending of light.

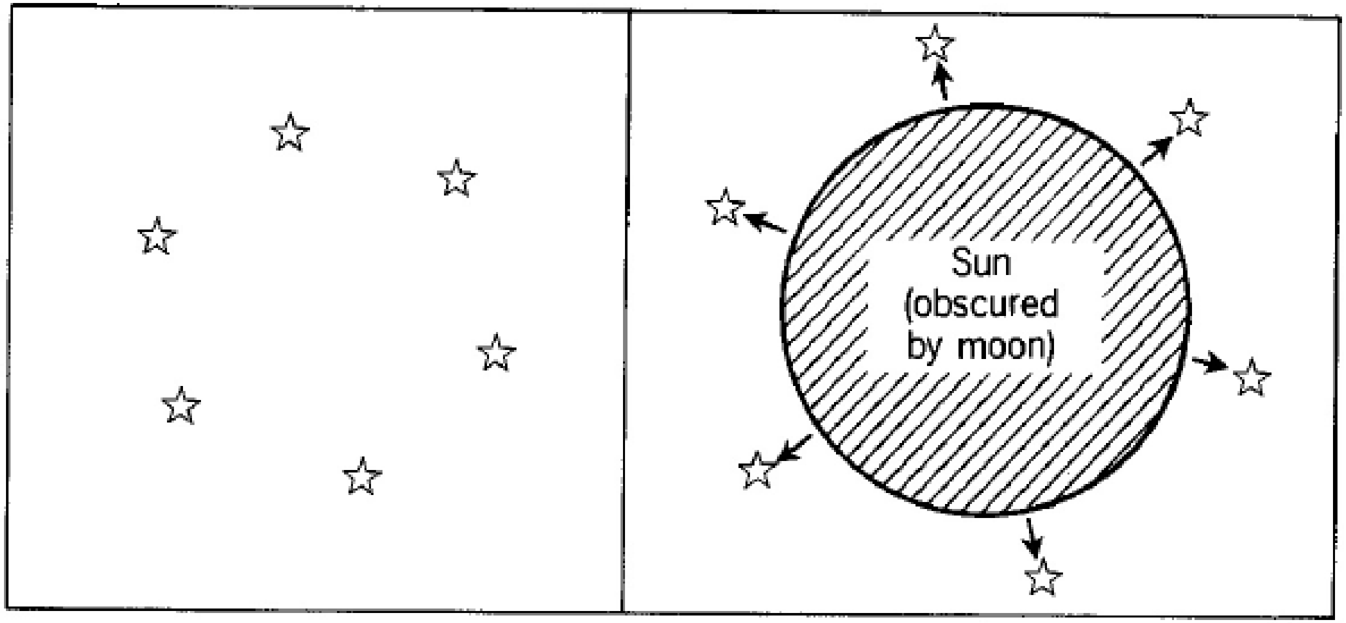


Figure : During Solar Eclipse, light from the Sun is blocked by Moon, consequently light from distant stars can be received on Earth. The evidence of spacetime curvature due to Sun is verified if the distant stars appear to be **shifted** to new locations during solar eclipse !!

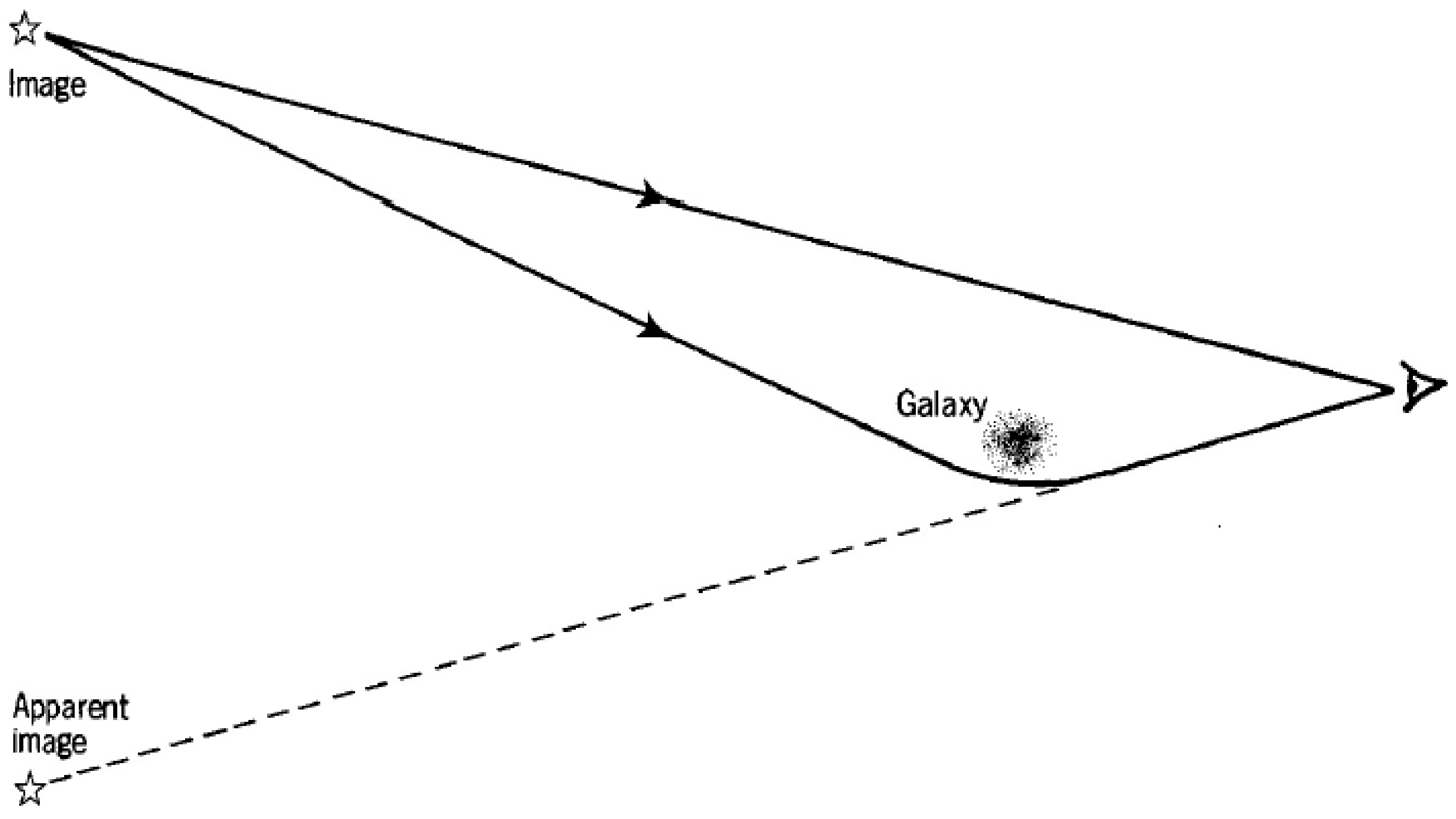


Figure : The "apparent image" of a "real" star is observed by projecting the light on the sky tangent to the light trajectory.

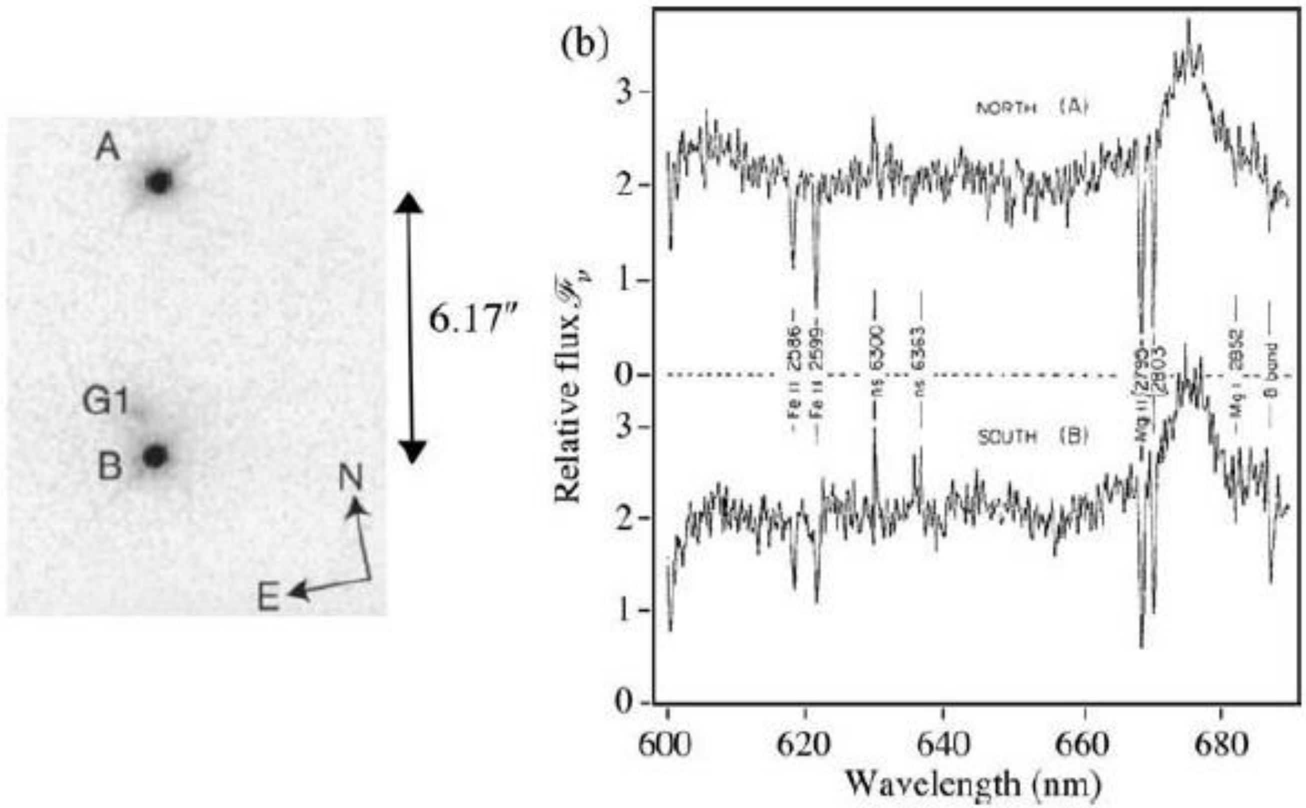


Figure : **Discovery of a "Double" quasar in 1979:** Twin images of a single quasar separated by 5.7 arc-seconds at redshift $z_s = 1.405$.

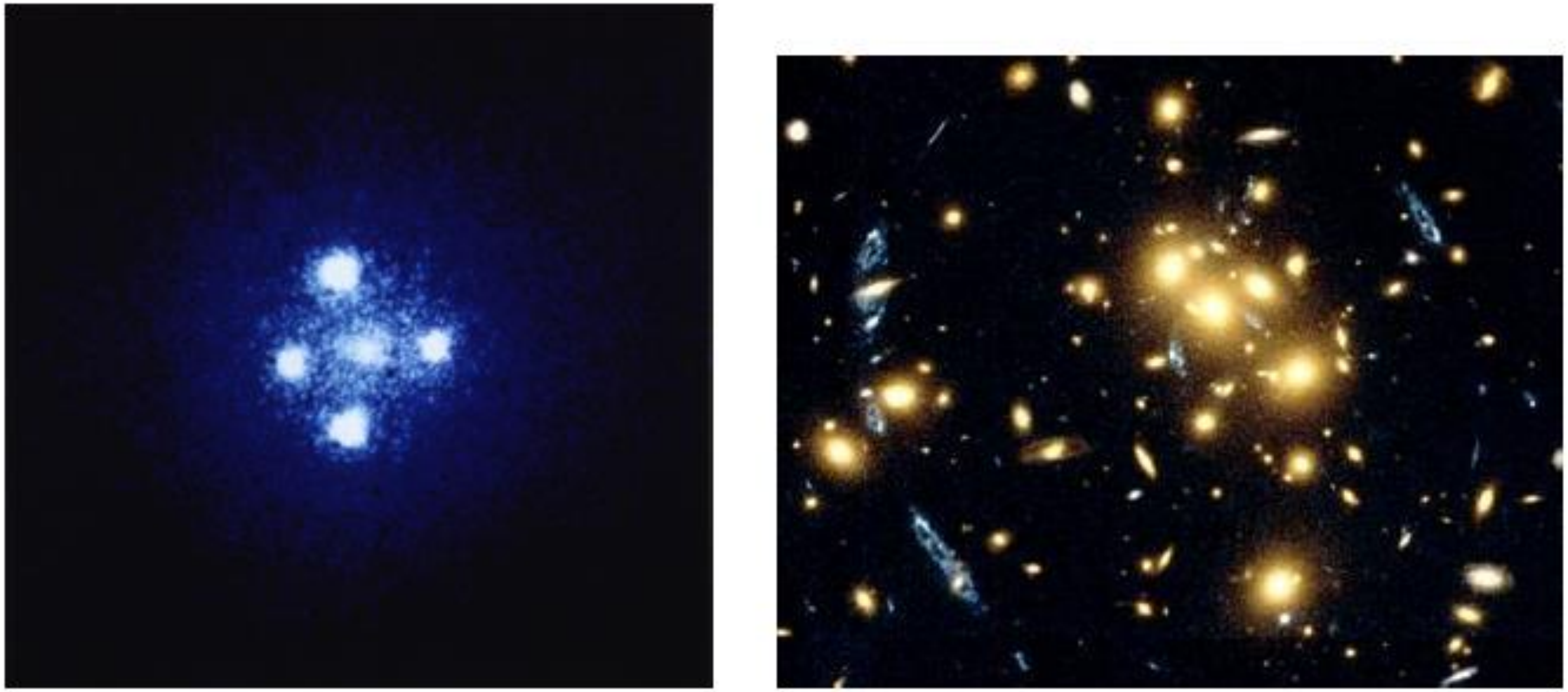
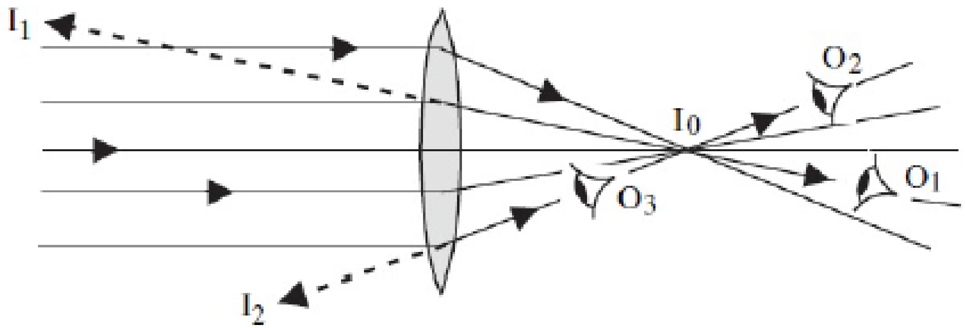


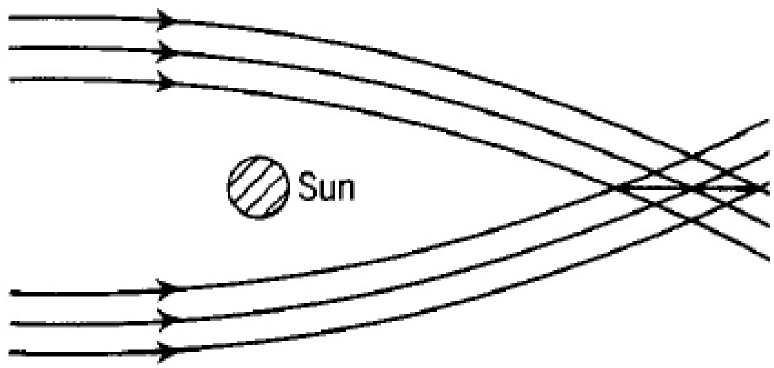
Figure : Generally one sees multiple images of distant galaxies/quasars due to lensing effect (**Macro-lensing**). (*Left*) **Einstein Cross**: Four images surrounding a lens galaxy. (*Right*) Formation of **Giant Arcs** due to lensing by galactic cluster.

An Ideal Convex Lens



- All the light rays from a distance source are converged to a Focus. All observers O_1 O_2 and O_3 perceive the source in the wrong directions !
- Only the observer at I_0 can identify the true location of the source as he receives maximum number of light rays compared to other observers.
- Light rays farther from the lens center bend with large deflection angle while those passing near the center bend with the least deflection angle.

A Gravitational Lens



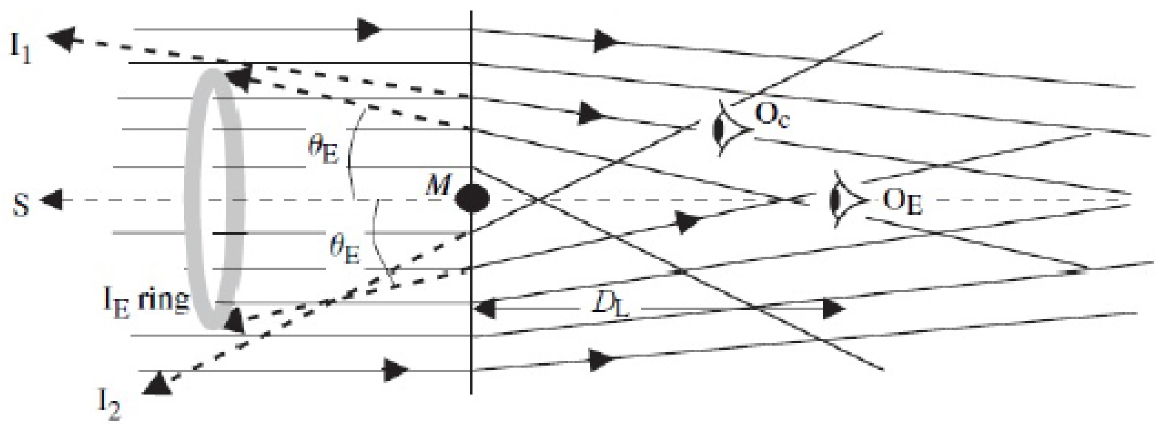


Figure : A Gravitational Lens does not possess a unique focus. Consequently it is impossible to get a clear image of a distant source. The Einstein Ring is formed when the troika (source, lens and observer) lie in the same line.

Figure : **Einstein Ring**

- The lens appears as a bright object (galaxy) in the center while the distant source (galaxy) appears as a ring due to perfect alignment.
- The radius/size of Einstein ring gives a measure of the spacetime curvature produced by the lens.

Bending of Light by a Schwarzschild Black Hole

Consider a Schwarzschild black hole of mass M, represented by the line element

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (1)$$

The corresponding Lagrangian ($\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$) is given by

$$\mathcal{L} = \left(1 - \frac{2GM}{c^2 r}\right)c^2 \dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2, \qquad (2)$$

where we assumed $\theta = \pi/2$ (equatorial plane of the black hole) and dot represents differentiation w.r.t λ. Since both t and ϕ are cyclic coordinates, the Euler-Lagrange equations give

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0 \quad \Rightarrow \quad r^2\dot{\phi} = \tilde{L} \equiv L/m, \qquad (3)$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = 0 \quad \Rightarrow \quad \left(1 - \frac{2GM}{c^2 r}\right)\dot{t} = \tilde{E} \equiv E/m, \qquad (4)$$

where $\tilde{L}$ and $\tilde{E}$ are respectively angular momentum/unit mass and energy/unit mass.

The trajectory of a light ray in the equatorial plane of a Schwarzschild black hole is obtained via $u^\mu u_\mu = 0$:

$$\left(\frac{dr}{d\lambda}\right)^2 = \tilde{E}^2 - \frac{\tilde{L}^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right). \tag{5}$$

Considering $r(\lambda) = \frac{1}{u(\lambda)}$ in (5), we get

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - u^2\left(1 - \frac{2GM}{c^2}u\right). \tag{6}$$

In the limit $\tilde{E} \to \infty$, $\tilde{L} \to \infty$, we get

$$\frac{1}{b^2} - \left(\frac{du}{d\phi}\right)^2 - u^2\left(1 - \frac{2GM}{c^2}u\right) = 0. \tag{7}$$

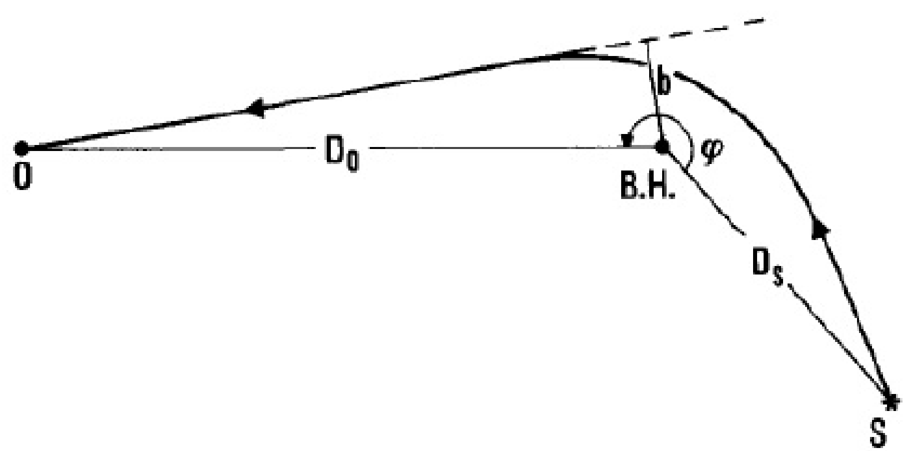
where we have used $\frac{1}{b^2} = \frac{\tilde{E}^2}{\tilde{L}^2}$ (Hehl & Frolov,2003).

At the closest approach to M, we have $\phi = \phi_m$, $r = r_m$; $\frac{dr}{d\phi}|_{r_m} = 0$, thus

$$L = \frac{r_m}{\sqrt{1 - \frac{2GM}{c^2 r_m}}}. \tag{8}$$

By performing change of variables $x = \frac{r_m}{r}$, we get the Einstein deflection angle for a Schwarzschild black hole is

$$\phi_m - \phi_\infty = \int_\infty^{r_m} \frac{dr}{r^2\sqrt{\frac{1}{r_m^2}\left(1 - \frac{2GM}{c^2 r_m}\right) - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)}} \tag{9}$$



Astrophysical Evidence of Black Holes

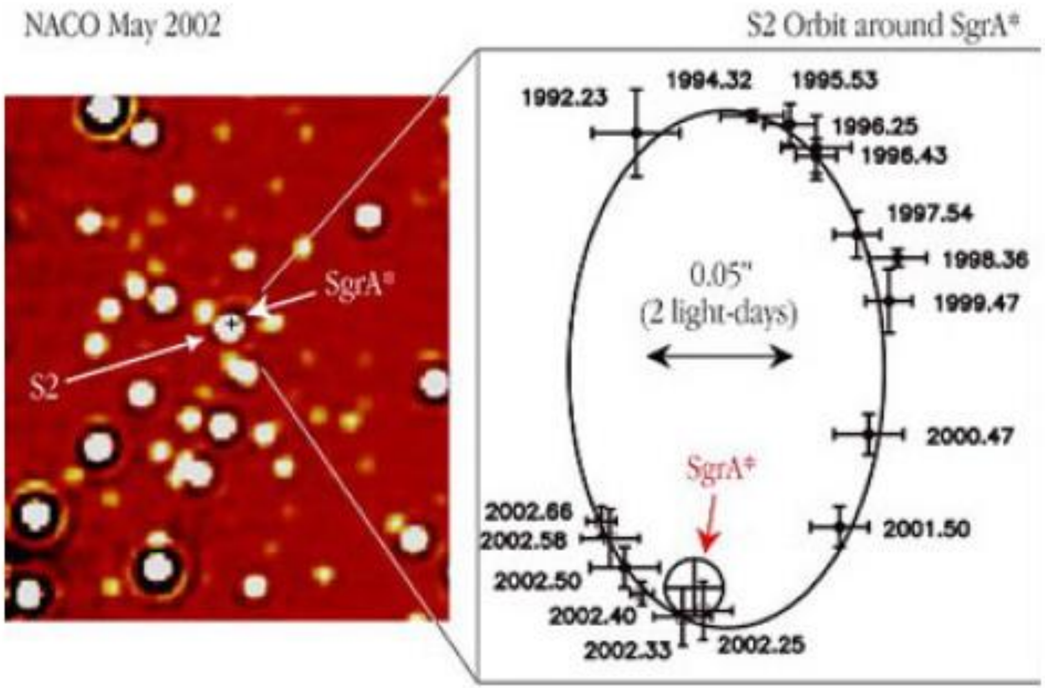


Fig.1

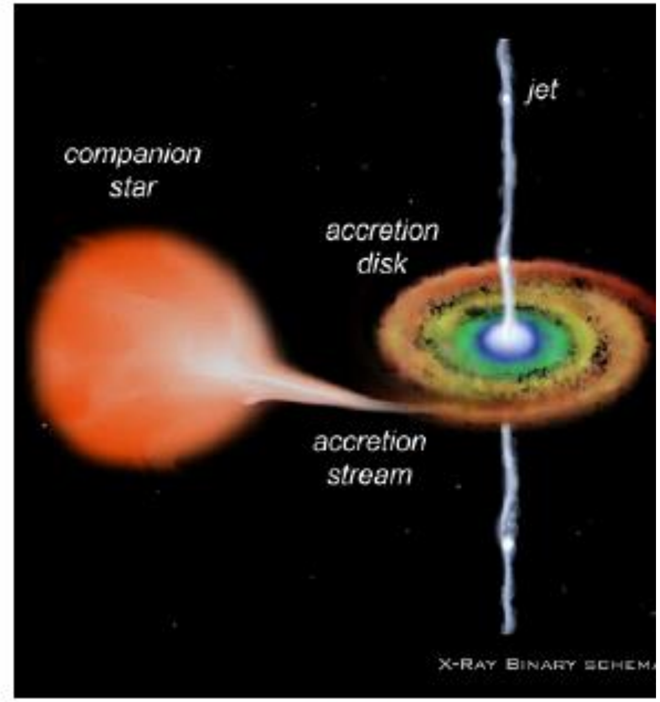


Fig.2

Figure : (Fig.1) A supermassive BH ($\sim 3.6 \times 10^6$ solar mass) is located in the center of Milky Way galaxy. (Fig.2) A typical X-Ray Binary containing a giant star and a solar mass BH, e.g. Cygnus X-1 $\sim$ 20 solar mass BH, (Melia,2007)

Effective Potential for Photons near a Schwarzschild Black Hole

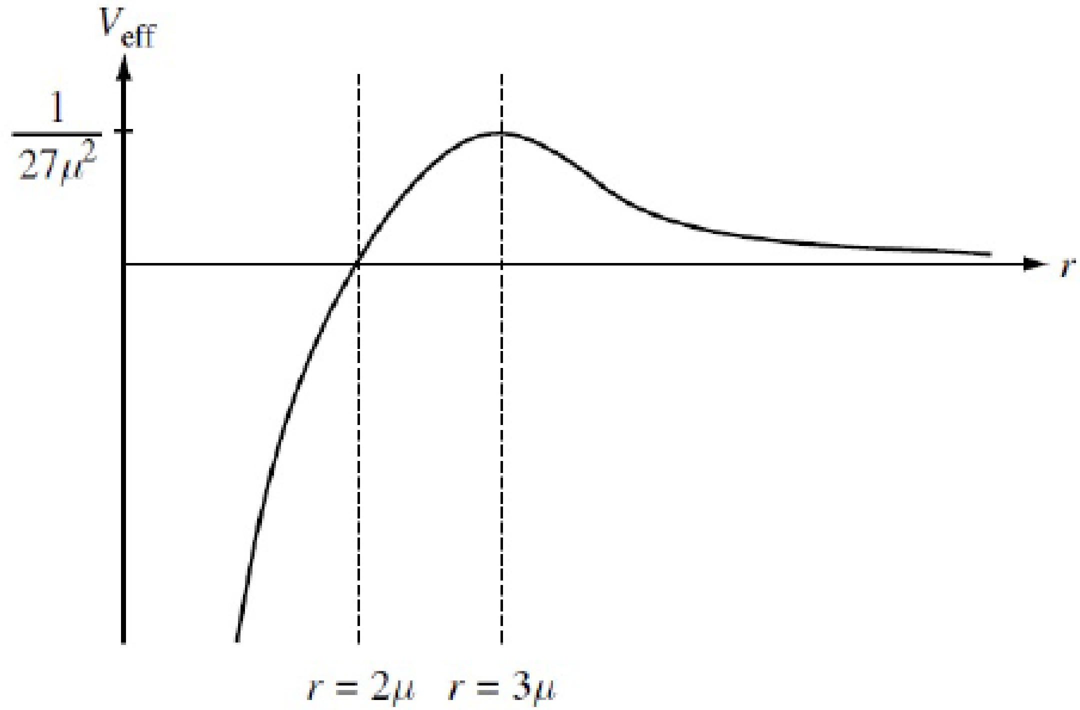
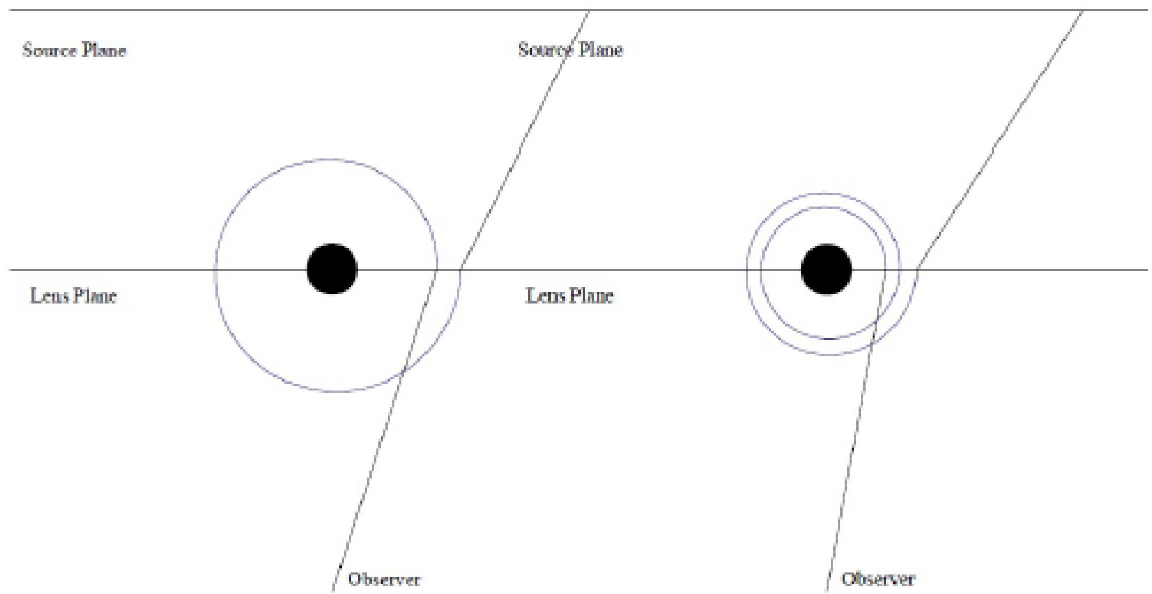


Figure : Photons can reach closest to the Schwarzschild black hole by $r_0 = 3\mu$ and orbit circularly (where μ is the mass of BH). Since r_0 is an unstable orbit ($V_{\rm eff}(r) = (1 - \frac{2\mu}{r})\frac{L^2}{r^2}$, $V''_{\rm eff}(r_0) < 0$), photons can escape from this orbit and reach the observer.



- A deflection by an angle π (or odd multiples of π) leads to reverse deflection i.e. back to source.
- A deflection by an angle 2π (or even multiples of π) leads to multiple images (ring like) . Each loop leads to two images formation.
- All images are merged in the case of perfect alignment between source, lens & observer leading to Einstein Ring.
- Photons traveling less than the photon sphere radius are trapped by black hole and are lost forever.

The magnification $\mathcal{M}$ of image is given by

$$\mathcal{M} \equiv \frac{\text{Solid Angle of the IMAGE}}{\text{Solid angle of the SOURCE}} = \frac{\sin\theta d\theta}{\sin\beta d\beta}. \tag{11}$$

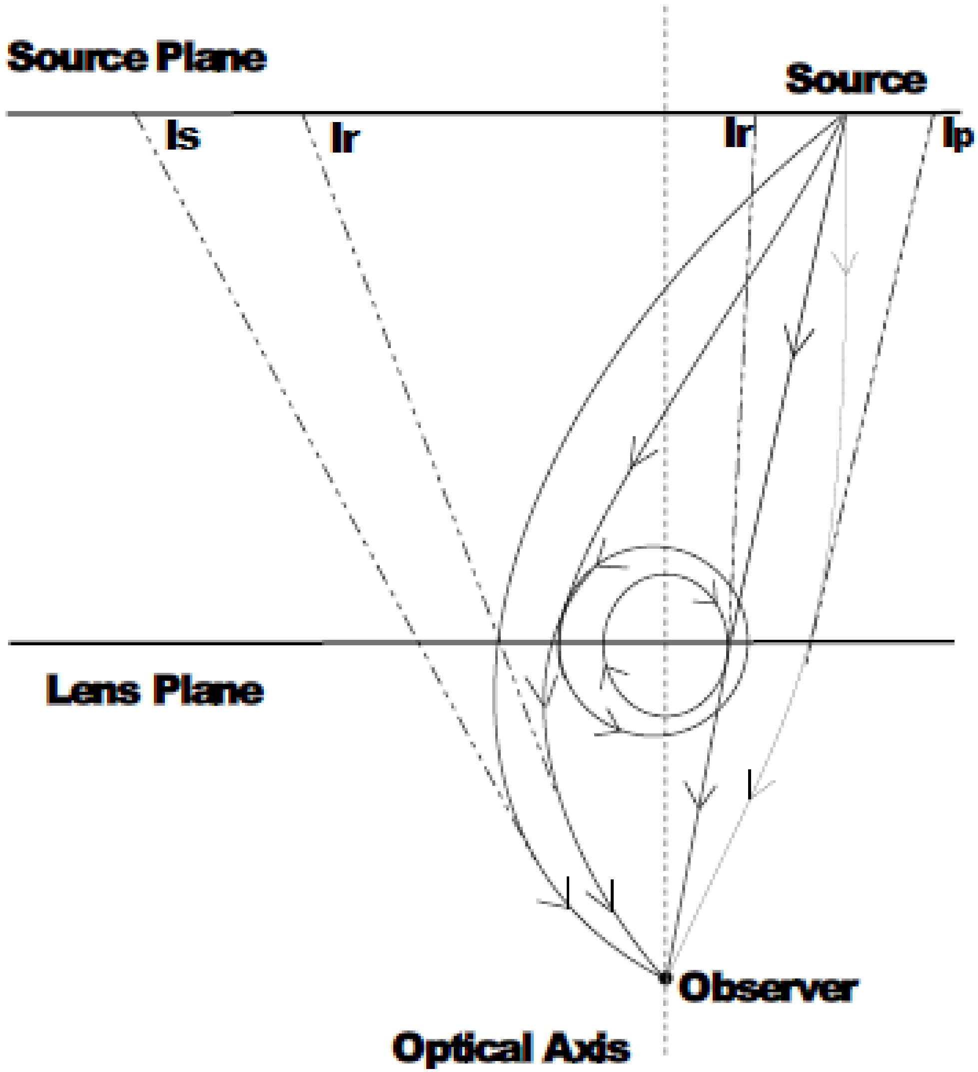
The tangential and radial magnifications are respectively given by

$$\mathcal{M}_t \equiv \frac{\sin\theta}{\sin\beta}, \quad \mathcal{M}_r \equiv \frac{d\theta}{d\beta}. \tag{12}$$

Hence

$$\mathcal{M} = \mathcal{M}_t \mathcal{M}_r.$$

Note that $\mathcal{M}$ decreases when β increases. Also $\mathcal{M}$ diverges at $\beta = 0$.


Source Plane
Source
Is
Ir
Ir
Ip
Lens Plane
Observer
Optical Axis

Gravitational Lensing by a Kiselev Black Hole

$$ds^2 = c^2 f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2,$$

$$f(r) = 1 - \frac{2GM}{c^2 r} - \frac{\sigma}{r^{3w_q+1}}.$$

Here, we also take $G = 1 = c$.

For $f(r) = 0$, we get two values of r, namely

$$r_+ = \frac{1 + \sqrt{1 - 8M\sigma}}{2\sigma}, \quad r_- = \frac{1 - \sqrt{1 - 8M\sigma}}{2\sigma}.$$

The Lagrangian for a photon traveling in Kiselev spacetime is given by

$$\mathcal{L} = (1 - \frac{2M}{r} - \sigma r)\dot{t}^2 - \frac{1}{1 - \frac{2M}{r} - \sigma r}\dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2.$$

Taking $(\theta = \frac{\pi}{2})$.

$$\mathcal{L} = (1 - \frac{2M}{r} - \sigma r)\dot{t}^2 - \frac{1}{1 - \frac{2M}{r} - \sigma r}\dot{r}^2 - r^2\dot{\phi}^2.$$

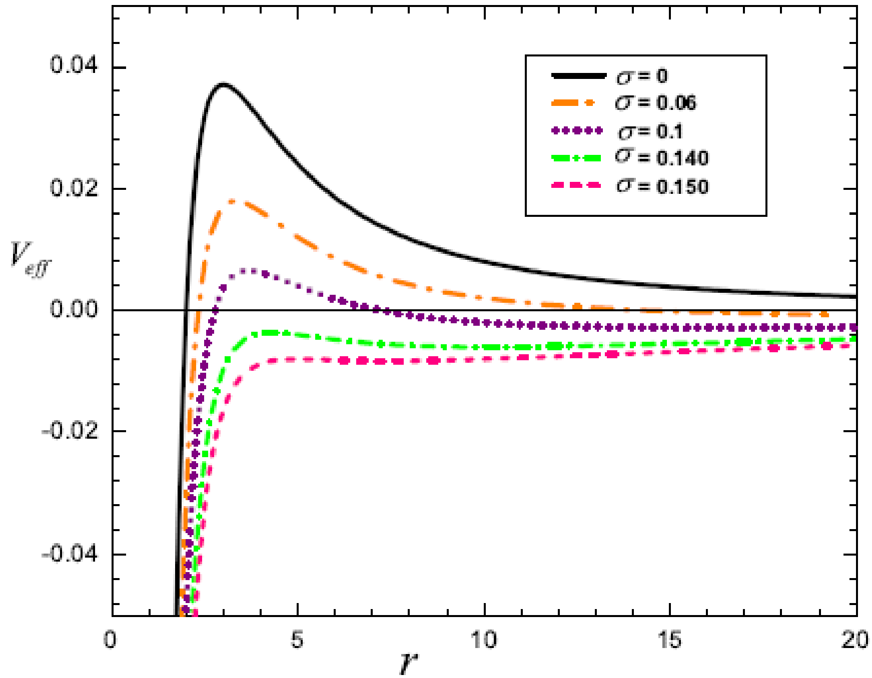
V. V. Kiselev, Class. Quan. Grav. 20, 1187 (2003).

Using the normalization condition of the 4-velocity $g_{\mu\nu}U^{\mu}U^{\nu} = 0$, we get the equation of motion for photons

$$\dot{r} = L\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r} - \sigma r\right)}, \quad \text{where} \quad b = |\frac{L}{E}|.$$

Here, the effective potential for photons is given by

$$V_{\text{eff}} = \frac{L}{r^2}\left(1 - \frac{2M}{r} - \sigma r\right).$$



To find the radius of circular orbit of photons, we use the condition $\frac{dV_{\rm eff}}{dr} = 0$ to obtain

$$r_{c\pm} = \frac{1 \pm \sqrt{1 - 6M\sigma}}{\sigma}.$$

Here r_{c+} is greater than the outer horizon r_+ while r_{c-} lies between inner and outer horizon ($r_- < r_{c-} < r_+$). The region of interest is between the horizons. Therefore, the radius of unstable circular orbit for photon is r_{c-} which is also called the photon sphere

$$r_{\rm ps} = \frac{1 - \sqrt{1 - 6M\sigma}}{\sigma}.$$

$\sigma \to 0$, we get the radius of photon sphere for SBH as

$$r^s_{\rm ps} = 3M.$$

The equation of path is

$$\left(\frac{du}{d\phi}\right)^2 - B(u) = 0,$$

where

$$B(u) = \frac{1}{b^2} - u^2\left(1 - 2Mu - \frac{\sigma}{u}\right).$$

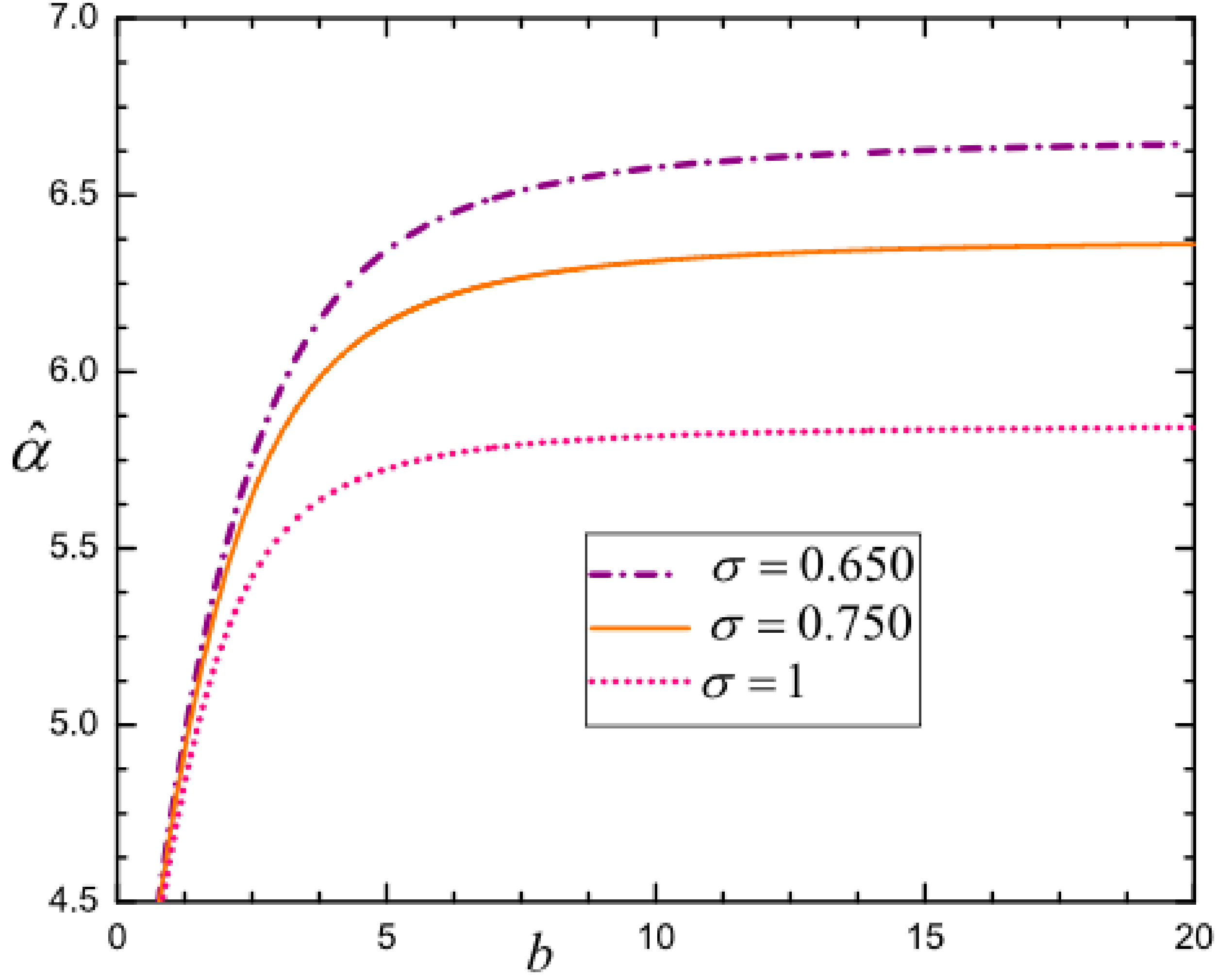
The critical value of the impact parameter is

$$b_{\rm sc} = \sqrt{\frac{r_{\rm ps}^3}{r_{\rm ps} - 2M - \sigma r_{\rm ps}^2}}.$$

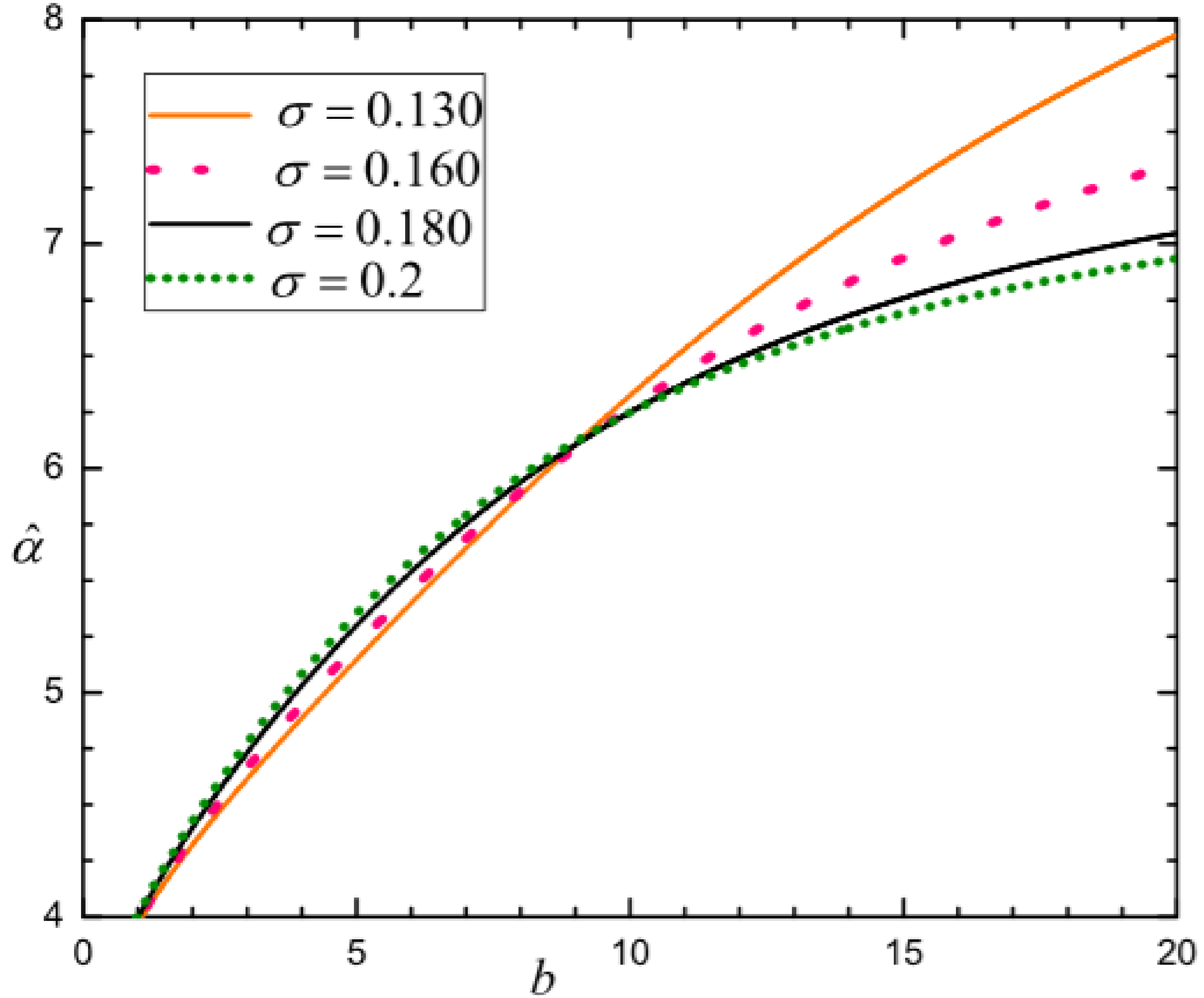
For a Schwarzschild black hole, it is

$$b_{\rm sc}^s = 3\sqrt{3}M.$$

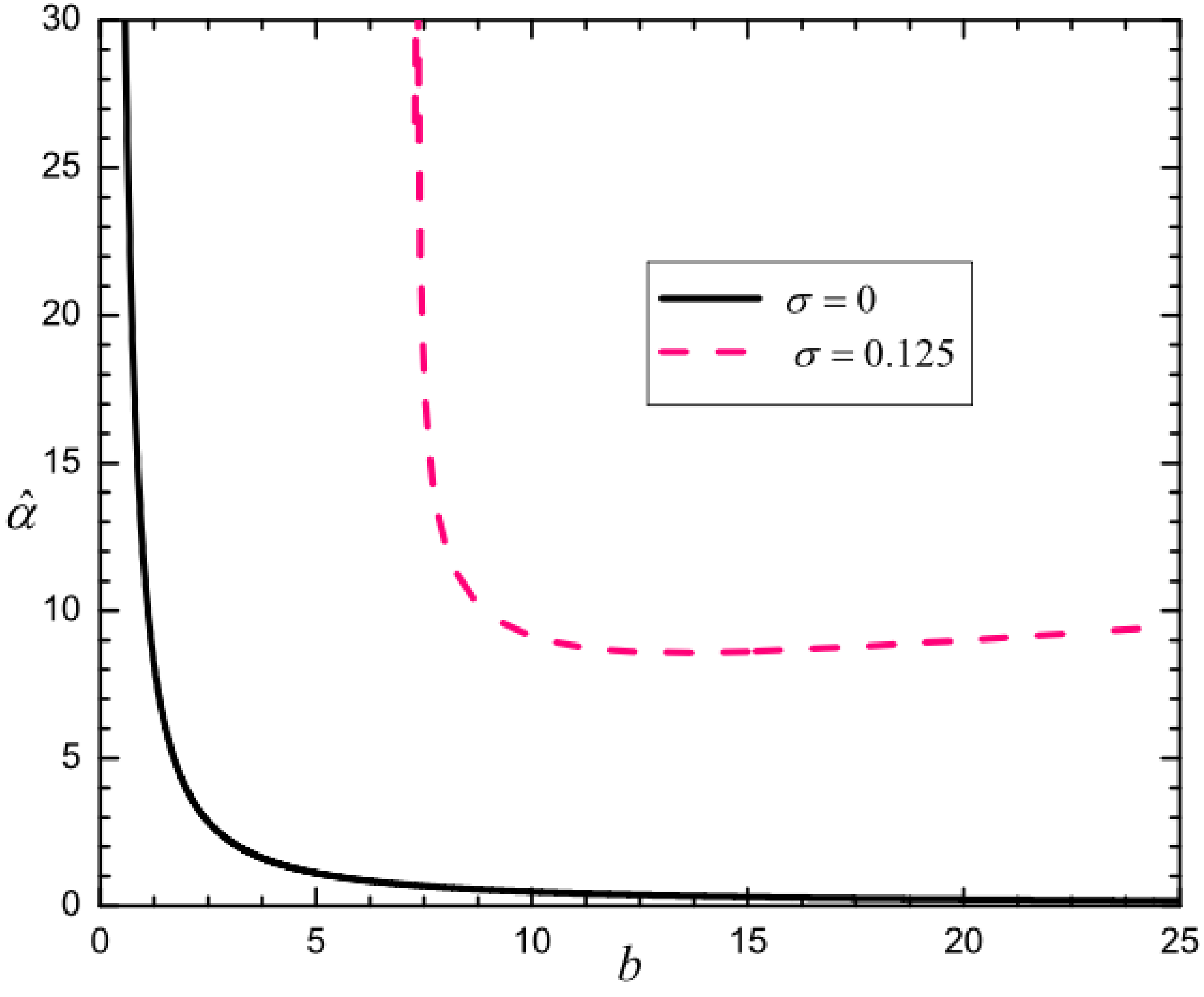
Bending Angle $\sigma > \frac{1}{8M}$



Naked Singularity at $\sigma > \frac{1}{8M}$



Bending Angle at $\sigma = \frac{1}{8M}$



Final Remarks

1. Gravitational lensing is a tool to study and analyze the properties of numerous astrophysial objects such as black holes, compact stars and galaxies.
2. In the case of black holes, the lensing effect gives rise to relativistic images and Einstein rings forming at the Einstein radius.
3. The bending angle of photons around black holes increases by increasing the impact parameter.
4. Observing the lensing phenomenon for black holes helps us understand the near horizon structure of spacetime.
5. One can analyze the lensing process for numerous black holes.

References:

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Mustapha Azreg-Aïnou, Sebastian Bahamonde, Mubasher Jamil
Journal-ref: Eur. Phys. J. C (2017) 77: 414

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Journal-ref: Phys. Rev. D 92, 084042 (2015)